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Convex Metric Spaces and Their Applications in Nonlinear Analysis

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Riham Zuhair Ali Alattas

Supervised by Professor Dr. Liaqat Ali Khan
Co-supervised by Dr. Saud Mastoor Alsulami

A Thesis submitted in partial fulfillment of the requirements for the Master's Degree in Science

Department of Mathematics, Faculty of Science
King Abdulaziz University, Jeddah, Saudi Arabia

Rajab 1434 A.H. - May 2013 A.D.

CERTIFICATE

"Non of the work referred to in this thesis has been submitted in support of an application for another degree at this or any other university or institution of learning."

Riham Zuhair Ali Alattas

DEDICATION

This work is dedicated with great respect and deep affection to my parents, my brothers, and my sisters.

ACKNOWLEDGEMENT

First and foremost thanks are to Allah (SWT) for giving me strength and guidance to complete this work.

I am greatly indebted to my parents for support during my work and research.

I would like to express my thanks to Dr. Liaqat Ali Khan for his valuable guidance during his supervision of preparing this thesis.

I would also like to greatly thank Dr. Abdullah Al-Otaibi (Chairman of the Department of mathematics), Dr. Muhammad Ali Alghamdi, Dr. Saud Alsulami, Dr. Hunaida Almalaika, Dr. Salma Altuwerqi, Dr. Fatima Kandil and other teachers of Mathematics Department at KAU for their encouragement and support.

May Allah (SWT) reward them abundantly.

TESTIMONY

This is to certify that the candidate **Riham Zuhair Ali Alattas** has worked under the supervision of the undersigned and completed this thesis to meet the partial requirement of a degree of Master of Science in Mathematics.

Professor Liaqat Ali Khan
Supervisor

ABSTRACT

Metric convexity is a fundamental concept in the study of geometry of metric spaces. There are several ways in which one can introduce the notion of convexity in metric spaces. In this thesis, we mainly consider W-convex metric spaces, introduced by Takahashi in 1970. Such spaces include Banach spaces, convex subsets of Banach spaces and certain subsets of metric linear spaces. Subsequently, Machado (1973), Talman (1977), and Naimpally, Singh and Whitfield (1983) and Beg and Azam (1986), among others, developed this theory and obtained fixed point theorems in W-convex metric spaces. Since then there has been great developments in this field and it is presently an active area of research. Further progress has recently been done in the related study of M-convex, Hyperconvex and CAT(0) spaces. In this thesis, we have presented an up-dated research done in this area and also its applications in Non-linear Analysis including Fixed Point Theory and Best Approximation.

INTRODUCTION

The notion of convexity plays an important role in many results in Mathematics. In particular, metric convexity is a fundamental concept in the study of geometry of metric spaces. There are several ways in which one can introduce the notion of convexity in metric spaces. Uniform convexity and strict convexity on Banach spaces were first introduced by Clarkson in 1936. These concepts were extended to metric linear spaces by Ahuja, Narang and Trehan in 1977. The hyperconvexity in general metric spaces (not necessarily linear) was introduced by Aronszajn and Panitchpakdi in 1956. The concept of M-convexity was introduced by Khalil in 1988.

A further different viewpoint to convexity is due to Takahashi (1970), who considers a metric space (X, d) to be convex if there exists a function $W : X \times X \times I \to X$, called a convex structure if, for any $(x, y, \lambda) \in X \times X \times I$ and $z \in X$,

$$d(z, W(x, y, t) \leq t.d(z, x) + (1 - t).d(z, y).$$

The set X with this convex structure W is called a **W-convex metric space** (or a **WCM space**) and is denoted by (X, d, W). Such spaces include Banach spaces, convex subsets of Banach spaces and certain subsets of metric linear spaces. Subsequently, Machado (1973), Talman (1977), and Naimpally, Singh and Whitfield (1983) and Beg and Azam (1986), among others, developed this theory and obtained fixed point theorems. Since then there has been great developments in this field and it is presently an active area of research. Further progress has recently been done in the related study of hyperconvex and CAT(0) spaces (see the list of references).

In this thesis, we have presented an exposition of the results of these authors on Fixed Point Theory and Best Approximation and obtain some generalized results.

In Chapter 1, we give basic definitions and results of Functional Analysis. These include Topological Spaces, Normed Spaces, Weak Topologies, Hausdorff Metric, Strictly and Uniformly Convex Spaces and Best Approximation.

In Chapter 2, we present some Classical Fixed Point Theorems and some of their interesting generalizations. These include famous results of Banach, Brouwer, Tychonoff-Schauder, Kannan, Edelstein, Kirk and Caristi.

We have considered the notion of W-convex Metric space in Chapter 3. Here we also study some Fixed Point and Best Approximation results for WCM Spaces.

Chapter 4 is devoted to the study of various types of M-convex Metric Spaces. We discuss relationship between such spaces. Also we give some results on Fixed Points and Best Approximation in Strict M-Convex metric spaces.

Finally, in Chapter 5, we study the classes of Hyperconvex Space, $\mathbb{R}$-trees and

CAT(0)-spaces. As applications, we again present some versions of Fixed Point Theorems and Best Approximation in such spaces.

Chapter 1

Functional Analysis Background

In this chapter, we include the basic material for the thesis such as (1) Topological and Metric Spaces, (2) Normed Spaces and Topological Vector Spaces, (3) Weak Topologies on Normed Spaces, and (4) Hausdorff Metric on $cb(X)$, (5) Strictly Convex and Uniformly Convex Metric Linear Spaces, (6) Convexity of Open balls in Metric Linear Spaces, (7) Best Approximations in Metric Linear Spaces. The detail of these can be found in the books of G.F. Simmons (1963), S. Willard (1970), W. Rudin (1991), E.W. Chenny (1966), I. Singer (1970), R.B. Holmes (1972), F. Deutsch (2001) and M.A. Khamsi and W.A. Kirk (2001) and some relevant papers.

1.1 Topological and Metric Spaces

In this section we present some basic definitions and results on topological spaces.

Definition: Let X be a non-empty set. A collection τ of subsets of X is called a **topology** on X if it has the following properties:

(i) X, $\varnothing \in \tau$.

(ii) The union of any number of members of τ belongs to τ.

(iii) The intersection of a finite number of members of τ belongs to τ.

In this case (X, τ) is called a **topological space** and each set U in τ is called an **open set**. A subset A of X is said to be **closed set** if its complement $X \backslash A$ is an open set. For any $x \in X$, a subset W of X is called a **neighborhood** of x if there exists an open set V such that $x \in V \subseteq W$. In particular, every open set U containing x is called an open neighborhood of x.

Definition: Let (X, τ) be a TS.

(1) For any $x \in X$, a subcollection $\beta_x \subseteq \tau$ is called a **local base** at x if, for each neighborhood U of x, there exists a $V \in \beta_x$ such that $x \in V \subseteq U$.

(2) A subcollection $\beta \subseteq \tau$ is called a **base** (or **basis**) for τ if, for any $U \in \tau$ and

$x \in U$, there exists $V \in \beta$ such that $x \in V \subseteq U$ (or equivalently $U \in \tau$ is a union of sets in β).

(3) A subcollection $\gamma \subseteq \tau$ is called a **subbase** (or **subbasis**) for τ if the finite intersection of sets in γ form a base of τ.

Definition: Let (X, τ) be a topological space and $A \subseteq X$. A point $x \in X$ is called a limit point of A if each neighborhood of x contains a point of A other than x. The closure of A, denoted by $\overline{A}$ or (*or cl*(A)), is defined as $\overline{A} = A \cup \{$all limit points of $A\}$. A is said to be dense in X if $\overline{A} = X$.

Definition: (1) A relation $\geq$ on a set D is called a **direction** (or a **directed relation**) if:

(i) $\alpha \geq \alpha$,for all $\alpha \in D$ (reflexive),

(ii) if $\alpha \geq \beta$, $\beta \geq \gamma$, then $\alpha \geq \gamma$ (transitive);

(iii) if $\alpha, \beta \in D$, there exists a $\gamma \in D$ such that $\gamma \geq \alpha$, $\gamma \geq \beta$ (directive).

In this case, $(D, \geq)$ is called a **directed set.**

(2) A **net** in a set X is a mapping f of a directed set D into X. We denote this net by $\{f(\alpha) : \alpha \in D\}$ or $\{x_\alpha : \alpha \in D\}$, where $x_\alpha = f(\alpha) \in X$.

(3) A net $\{x_\alpha : \alpha \in D\}$ in a topological space (X, τ) is said to be **convergent** to $x \in X$ if, given any neighborhood U of x, there exists $\alpha_0 \in D$ such that

$$x_\alpha \in U \quad \text{for all } \alpha \geq \alpha_0,$$

(i.e., $x_\alpha \in U$ eventually). In this case, x is called the limit of the net and we write $x_\alpha \longrightarrow x$.

Definition: (1) If X is a non empty set, then a function $d : X \times X \to \mathbb{R}$ is called a **metric** on X if, for any $x, y, z \in X$,

(M_1) $d(x, y) \geq 0$;

(M_2)(i) $d(x, x) = 0$, i.e. $x = y \Rightarrow d(x, y) = 0$;

(M_2)(ii) $d(x, y) = 0 \Rightarrow x = y$;

(M_3) $d(x, y) = d(y, x)$ (symmetry);

(M_4) $d(x, y) \leq d(x, z) + d(z, y)$ (triangle inequality).

(2) If d is a metric on a set X, then the pair (X, d) is called a **metric space**.

(3) For any $x \in X$ and $r > 0$, the sets

$$\begin{aligned} B(x, r) &= \{y \in X : d(y, x) < r\} \\ B[x, r] &= \{y \in X : d(y, x) \leq r\} \end{aligned}$$

are called **open ball** and **closed ball**, respectively, with center x and radius r.

(4) A subset U of a metric space (X, d) is called a d**-open set** if, for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$.

(5) If d is a metric on a set X, then collection t_d of all d-open sets is a topology on X, called the **metric** induced by d. Clearly, the collection $\mathcal{B} = \{B(x,r) : x \in X, r > 0\}$ is a base for the topology t_d.

(3) A topological space (X,τ) is said to be **metrizable** if there exists a metric (resp. pseudo-metric) d on X such that $\tau = t_d$.

Note. (1) Any non-empty set X becomes a metric space with respect to the **trivial metric** given by

$$d(x,y) = \begin{cases} 1 & \text{if} \quad x \neq y \\ 0 & \text{if} \quad x = y, \end{cases} \quad x, y \in X.$$

(2) If (X,τ) is a topological space, then τ is the discrete topology iff it is induced by the trivial metric.

Now we consider various generalization of the metric function ([BKMP09], [BBI01]).

Definition: For any set X, a function $d : X \times X \to \mathbb{R}$ is called:

(i) a **semi-metric** (or **pseudo-metric**) on X if it satisfies all conditions of a metric except (M_2)(ii).

(ii) an **asym-metric** on X if it satisfies all conditions of a metric except (M_3) (symmetry).

(iii) a **quasi-metric** on X if it satisfies all conditions of a metric except (M_4) (triangular inequality).

(iv) a **pre-metric** on X if it satisfies all conditions of a metric except (M_2)(ii), (M_3) and (M_4).

Example (1) For any set X (with two or more distainct points), define $d : X \times X \to \mathbb{R}$ by

$$d(x,y) = 0 \text{ for all } x, y \in X.$$

Then d is a semi-metric but not a metric (since for $x \neq y \in X$, $d(x,y) = 0$). Clearly, d is a metric on X iff X is a singleton.

Example (2) If $X = \mathbb{R}$, let $d : \mathbb{R} \times \mathbb{R} \to \mathbb{R}$ be given by

$$d(x,y) = \begin{cases} x - y & \text{if} \quad x \geq y \\ 1 & \text{if} \quad x < y, \end{cases} \quad x, y \in \mathbb{R}.$$

Then d is an asym-metric, but not a metric on $\mathbb{R}$ (since, for $x > y$, $d(x,y) = x - y \neq y - x = d(y,x)$ and so d is not symmetric).

Example (3) For any metric space (X,d), define $\gamma : 2^X \to \mathbb{R}$ by

$$\gamma(A,B) = \inf\{d(x,y) : x \in A, y \in B\},\ A, B \in 2^X.$$

Then γ is a pre-metric on 2^X but it need not be semi-metric, asym-metric or quasi-metric (since it need not satisfy(M_2)(ii), (M_3) and (M_4)).

Remarks. (1) If d is a asym-metric on any set X, then the function $d^* : X \times X \to \mathbb{R}$ defined by

$$d^*(x, y) = \frac{1}{2}[d(x, y) + d(y, x)], \text{ for all } x, y \in X,$$

is symmetric and hence a metric on X.

(2) As in the case of a metric space, a pre-metric d on a set X gives rise to a topology as follows: A subset U of (X, d) is called a **d-open set** if, for each $x \in U$, there exists $r > 0$ such that $B(x, r) \subseteq U$. However, in this case, in general, the d-open balls $B(x, r)$ themselves need not be open sets with respect to this topology (as its proof requires the triangle inequality and symmetry). In fact, the interior of a d-open ball $B(x, r)$ may be empty.

Definition: [BKMP09] A function $d : X \times X \to \mathbb{R}$ is called a **partial metric** on X if, for any $x, y, z \in X$,

(PM_1) $d(x, y) \geq 0$;

(PM_2)(i) $x = y \Leftrightarrow d(x, x) = d(x, y) = d(y, y)$;

(PM_2)(ii) $d(x, x) \leq d(x, y)$;

(PM_3) $d(x, y) = d(y, x)$ (symmetry);

(PM_4) $d(x, y) \leq d(x, z) + d(z, y) - d(z, z)$.

Example. If $X = \mathbb{R}^+$, let $d : \mathbb{R}^+ \times \mathbb{R}^+ \to \mathbb{R}$ be given by

$$d(x, y) = \max\{x, y\} \text{ for all } x, y \in \mathbb{R}^+.$$

Then d is a partial metric on X.

Definition: Let (X, d) be a metric space.

(1) A sequence $\{x_n\} = \{x_n : n \in \mathbb{N}\} \subseteq (X, d)$ is said to be **convergent** to $x \in X$ if, given any $\varepsilon > 0$, there exists an integer n_0 such that

$$d(x_n, x) < \varepsilon \text{ for all } n \geq n_0;$$

in this case, we write $x_n \to x$.

(2) $\{x_n\}$ is said to be a **Cauchy sequence** if, given any $\varepsilon > 0$, there exists an integer n_0 such that

$$d(x_n, x_m) < \varepsilon \text{ for all } n, m \geq n_0.$$

Note. Clearly, every convergent sequence is a Cauchy sequence.

Definition: Let A be a subset of a metric space (X, d).

(i) The **diameter** $\delta(A)$ of A is defined as

$$\delta(A) = \sup\{d(x, y) : x, y \in A\}.$$

(ii) A is called a **bounded set** if its diameter $\delta(A) < \infty$, i.e., if there exists a constant $M > 0$ such that

$$d(x,y) \leq M \text{ for all } x, y \in A;$$

Note. (1) It is easy to see that every Cauchy sequence (in particular, convergent sequence) $\{x_n\}$ in (X,d) is bounded.

(2) The converse need not hold. The sequence

$$\{x_n\} = \{(-1)^n : n \geq 1\} = \{-1, 1, -1, 1, ...\} \subseteq \mathbb{R}$$

is bounded but neither Cauchy nor convergent. However, it has two convergent subsequence

$$\begin{aligned} \{x_{2n}\} &= \{(-1)^{2n}\} = \{1, 1, 1, ...\} \to 1, \\ \{x_{2n-1}\} &= \{(-1)^{2n-1}\} = \{-1, -1, -1, ...\} \to -1. \end{aligned}$$

(3) (i) If $I = [a,b]$ or (a,b), then its diameter $\delta(I) = b - a$.

(ii) If $A = \{\frac{1}{n} : n \geq 1\} = \{1, \frac{1}{2}, \frac{1}{3}, ...\} \subseteq \mathbb{R}$, then

$$\delta(A) = \sup\{|\frac{1}{n} - \frac{1}{m}| : n, m \geq 1\} = 1.$$

(iii) If $A = \{1, 2, 3, ...\} \subseteq \mathbb{R}$, then $\delta(A) = \infty$.

(iv) The diameter of a triangle (or a triangular area) in $\mathbb{R}^2$ is the length of its longest side.

(v) The diameter of a rectangle (or a rectangular area) in $\mathbb{R}^2$ is the length of its diagonal.

Definition: Let (X,d) be a metric space, and let $A \subseteq X$. Then A is called **complete** if every Cauchy sequence in A converges to a point in A.

Note. Let $A \subseteq (X,d)$. Then:

(a) A is complete $\Rightarrow$ A is closed.

(b) Conversely, suppose (X,d) is complete. Then A is closed $\Rightarrow$ A is complete, (hence A is complete $\Leftrightarrow$ A is closed).

Definition: Let X and Y be a topological spaces.

(1) A function $f : X \to Y$ is said to be **continuous at** $x_0 \in X$ if, for each neighborhood V of $f(x_0)$ in Y, there exists a neighborhood U of x_0 in X such that $f(U) \subseteq V$; f is said to be **continuous on** X if it is continuous at each $x \in X$.

(2) A function $f : X \to Y$ is called a **homeomorphism** if f is continuous, one-one, onto and $f^{-1} : Y \to X$ is continuous.

Note. It is often convenient to use the following equivalent conditions for continuity of $f : X \longrightarrow Y$:

(a) For each open set $V \subseteq Y, f^{-1}(V)$ is open in X.

(b) For each closed set $W \subseteq Y$, $f^{-1}(W)$ is closed in X.

(c) For any $x \in X$. and a net $\{x_\alpha\} \subseteq X$,

$$x_\alpha \to x \text{ in } X \text{ implies that } f(x_\alpha) \to f(x) \text{ in } Y.$$

Definition: Let (X, d_X) and (Y, d_Y) be metric spaces. A mapping $f : (X, d_X) \longrightarrow (Y, d_Y)$ is said to be **continuous at** $x_0 \in X$ if, given any $\varepsilon > 0$, $\exists\ \delta > 0$ such that

$$d_Y(f(x), f(x_0)) < \varepsilon \ \text{ if } \ d_X(x, x_0) < \delta.$$

Theorem 1.1.1. *Let (X, d_X) and (Y, d_Y) be metric spaces and $f : (X, d_X) \longrightarrow (Y, d_Y)$ a function. Then the following conditions are equivalent:*

(a) *f is continuous at $x_0 \in X$.*

(b) *For any sequence $\{x_n\} \subseteq X$,*

$$\begin{aligned} x_n &\to x_0 \text{ in } X \text{ implies that } f(x_n) \to f(x_0) \text{ in} Y \\ \text{i,e., } \lim_{n\to\infty} f(x_n) &= f(\lim_{n\to\infty} x_n). \end{aligned}$$

Definition: A topological space (X, τ) is called:

(1) **Hausdorff** if, for any two points $x \neq y$ in X, there exist two open sets $U, V \subseteq X$ such that $x \in U$, $y \in V$ and $U \cap V = \varnothing$;

(2) **normal** if, for any two closed sets $A, B \subseteq X$ with $A \cap B = \varnothing$, there exist two open sets $U, V \subseteq X$ such that $A \subseteq U, B \subseteq V$ and $U \cap V = \varnothing$;

(3) **completely regular** if, for any closed set $A \subseteq X$ and $x \in X$ with $x \notin A$, there exists a continuous function $f : X \to [0, 1]$ such that $f(x) = 0$ and $f(A) = 1$ (i.e., $f(y) = 1$ for all $y \in A$).

Definition: Let (X, τ) be a topological space, and let $A \subseteq X$.

(1) A collection $U = \{G_\alpha : \alpha \in I\}$ of open subsets of X is said to be an **open cover** of A if

$$A \subseteq \cup_{\alpha\in I} G_\alpha.$$

If $\mathcal{U}$ *and* $\mathcal{V}$ are two covers of X, then $\mathcal{V}$ is said to be a **refinement** of $\mathcal{U}$ if each $V \in \mathcal{V}$ is contained in some $U \in \mathcal{U}$.

(2) A is called **compact** if every open cover of A has a finite subcover.

(3) A is called **sequentially compact** if every sequence $\{x_n\}$ in A has a convergent subsequence $\{x_{n_k}\}_{k=1}^{\infty}$ with limit in A.

(4) A is called **relatively compact** if its closure $\overline{A}$ is compact.

(5) X is called **locally compact** if each $x \in X$ has a relative compact neighbourhood;

(6) X is called **paracompact** if every open cover $\mathcal{U}$ of X has refinement $\mathcal{V}$ which is locally finite (i.e. each $x \in X$ has a neighbourhood which intersects only a finite number of members of $\mathcal{V}$);

(7) A collection $\{A_\alpha : \alpha \in I\}$ of subsets of X is said to have the **finite intersection property** if, for any finite subcollection $\{A_{\alpha_1}, ..., A_{\alpha n}\}$ of $\{A_\alpha : \alpha \in I\}$,

$$\bigcap_{i=1}^{n} A_{\alpha i} \neq \varnothing.$$

In this case, it may happen that $\bigcap\limits_{\alpha \in I} A_\alpha = \varnothing$.

Theorem 1.1.2. [Sim63; Will70] (a) *A closed subset A of a compact space X is compact.*

(b) *A compact subset A of a Hausdorff space X is closed.*

(c) *The continuous image of a compact set is compact.*

(d) *Every compact metric space is complete.*

(e) *Every metric space is normal, completely regular and paracompact.*

(f) *A metric space X is compact iff it is sequentially compact.*

(g) A topological space is compact iff, for any collection $\{A_\alpha : \alpha \in I\}$ of closed subsets of X with the finite intersection property, $\bigcap\limits_{\alpha \in I} A_\alpha \neq \varnothing$.

Note. As counter-examples, we mention that:

(1) $X = \mathbb{R}$ with the usual metric is not compact: The collection $\mathcal{U} = \{(-n, n) : n \in N\}$ is an open coner of $\mathbb{R}$, but it has no finite subcover of $\mathbb{R}$.

(2) Let $X = [0, 1]$ with the usual metric and $A = (0, 1)$. Then $A \subseteq X$ and X is compact, but its subset A is neither closed nor compact.

(3) Let $X = \mathbb{R}$ with the usual metric and $A_r = [r, \infty)$. Then $\{A_r : r \in \mathbb{R}\}$ is a collection of closed sets in $\mathbb{R}$ with the finite intersection property, but $\bigcap\limits_{r \in \mathbb{R}} A_r = \varnothing$; hence $\mathbb{R}$ is not compact.

(4) The set $\mathbb{R}$ of all real numbers with the usual topology given by the metric

$$d(x, y) = |x - y|, \quad x, y \in \mathbb{R},$$

is locally compact and hence completely regular, but is not compact.

(5) The set $\mathbb{Q}$ of all rational numbers with the usual topology is not locally compact; however, being a metric space, it is completely regular.

Definition: Let (X, d) be a metric space and $A \subseteq X$. Then A is called **totally bounded** (or **precompact**) if, for any $\varepsilon > 0$, there exists a finite set $\{x_1, x_2, ..., x_n\} \subseteq A$ such that

$$A \subseteq \cup_{i=1}^{n} B(x_i, \varepsilon).$$

Theorem 1.1.3. *Let (X, d) be a metric space and $A \subseteq X$. Then:*

(a) *A is compact iff every sequence $\{x_n\}_{n=1}^{\infty}$ in A has a convergent subsequence with limit in A.*

(b) *A is totally bounded iff every sequence $\{x_n\}_{n=1}^{\infty}$ in A has a Cauchy subsequence.*

(c) *A is compact $\Rightarrow$ A is totally bounded $\Rightarrow$ A is bounded; the converse need not hold.*

(d) *A is compact iff A is complete and totally bounded.*

(e) If *$A \subseteq \mathbb{R}^n$ or $\mathbb{C}^n$, then A is compact iff A is closed and bounded.*

Examples. (1) Let $X = \mathbb{R}$ with the usual metric and $A = (0, 1)$. Then A is totally bounded and bounded but not compact or complete.

(2) Let $X = \mathbb{R}$ with the discrete metric and $A = \mathbb{R}$. Then A is bounded but not totally bounded or compact.

Theorem 1.1.4. (Urysohn lemma) (a) *Let X be a normal space, and let A and B be closed subsets of X with $A \cap B = \varnothing$. Then there exists a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(A) = 0$ and $\varphi(B) = 1$.*

(b) *Let X be a locally compact Hausdorff space, K a compact subset of X, and U a neighborhood of K. Then there exists a continuous function $\varphi : X \to [0, 1]$ such that $\varphi(K) = 1$ and the support of φ is compact and contained in U. (The* **support** *of φ is the closure of the set $\{x \in X : \varphi(x) \neq 0\}$).*

Theorem 1.1.5. (Tietze extension theorem) *Let X be a normal space, and let A be a closed subset of X. Then, for any continuous function $f : A \to [0, 1]$ (or $\mathbb{R}$), there exists a continuous function $g : A \to [0, 1]$ (or R) such that $g = f$ on A. (Here g is called a continuous extension of f from A to X).*

Definition: If A is a bounded subset of $\mathbb{R}$, then:

(a) $\alpha = \inf A$ means: (i) $\alpha \leq x$ for all $x \in A$; (ii) given any $\varepsilon > 0$, there exists $y \in A$ such that $\alpha \leq y < \alpha + \varepsilon$.

(b) $\beta = \sup A$, means: (i) $x \leq \beta$ for all $x \in A$; (ii) given any $\varepsilon > 0$, there exists $z \in A$ such that $\beta - \varepsilon < z \leq \beta$

Remark. If A is a bounded subset of $\mathbb{R}$, $\alpha = \inf A$ and $\beta = \sup A$, then there exist sequences $\{a_n\}$, $\{b_n\}$ in A such that $a_n \to \alpha$ and $b_n \to \beta$. Hence $\alpha, \beta \in \overline{A}$.

Definition: (i) A function $\varphi : X \to \mathbb{R}$ is **continuous at** $x_0 \in X$ if

$$\lim_{x \to x_0} \varphi(x) = \varphi(x_0).$$

(ii) A function $\varphi : X \to \mathbb{R}$ is **upper continuous at** $x_0 \in X$ if

$$\limsup \varphi(x) \leq \varphi(x_0) \text{ as } x \to x_0.$$

(iii) A function $\varphi : X \to \mathbb{R}$ is **lower continuous at** $x_0 \in X$ if

$$\liminf \varphi(x) \geq \varphi(x_0) \text{ as } x \to x_0.$$

Theorem 1.1.8. (a) *A function $\varphi : X \to \mathbb{R}$ is continuous $\Leftrightarrow$ it is both upper and lower semicontinuous.*

(b) *The characteristic function χ_A of a subset A of X is upper (resp. lower) semicontinuous $\Leftrightarrow$ A is closed (resp. open).*

(c) *Every upper (resp. lower) semicontinuous function assumes its supremum (resp. infimum) on a compact set. In particular, every non-negative upper semicontinuous function on a compact set is bounded.*

Theorem 1.1.9. (Dieudonne Interpolation Theorem). *Let X be a paracompact Hausdorff space, $f : X \to \mathbb{R}$ a lower semicontinuous and $g : X \to \mathbb{R}$ an upper semicontinuous with $f \leq g$. Then there is an $h \in C_b(X, \mathbb{R})$ such that*

$$f(x) \leq h(x) \leq g(x) \text{ for all } x \in X.$$

Proof. [Wil70, p. 159] (Outline) For each rational r, let

$$G_r = \{x \in X : f(x) < r < g(x)\}.$$

Since X is paracompact, we can choose locally finite open coverings $\{U_r : r \in \mathbb{Q}\}$ and $\{V_r : r \in \mathbb{Q}\}$ of X such that

$$\overline{V_r} \subseteq U_r \subseteq G_r.$$

Define $h_r : X \to [-\infty, \infty]$ by

$$h_r(x) = \begin{cases} -\infty & \text{if } x \in X \backslash U_r, \\ r & \text{if } x \in \overline{V_r}. \end{cases}$$

Let

$$h(x) = \inf\{h_r(x) : r \in \mathbb{Q}\} \quad x \in X.$$

Then, as in the proof of Urysohn's Lemma, f is continuous on X. Further, $f(x) \leq h(x) \leq g(x)$ for all $x \in X$. $\square$

Note. The above theorem is interesting and famous that if $f \leq g$ which are not necessarily continuous, but there exists a continuous function h such that $f \leq h \leq g$.

Retracts of Topological Spaces

Definition: ([KK01], p. 176) Let X be a topological space and $A \subseteq X$. Then A is called a **retract** of X if there exists a continuous function $r : X \to A$ such that $r(x) = x$ for all $x \in A$. r is called a **retraction** of X into A.

Note. (1) Every retraction $r : X \to A$ is onto; i.e., $r(X) = A$: [Since $r : X \to A$, clearly $r(X) \subseteq A$. On the other hand, $A = r(A) \subseteq r(X)$ since $A \subseteq X$. Hence $r(X) = A$.]

(2) A retraction $r : X \to A$ need not be one-one: [Suppose $A \neq X$, and choose $x \in X \backslash A$. Then $b = r(x) \in A$. So $b \neq x$ (because $b \in A$ and $x \in X \backslash A$), but $r(b) = b = r(x)$. Hence r is not one-one.]

Note. (1) Let X be a topological space. Then:

(i) X is a retract of X.

(ii) For each $x_0 \in X, \{x_0\}$ is a retract of X.

Proof. (i) Clearly, the identity map $i : X \to X$, $i(x) = x \ \ (x \in X)$ is a retraction.

(ii) Clearly, the constant mapping $r : X \to \{x_0\}$,$r(x) = x_0 \ \ (x \in X)$ is continuous. Further, $r(x) = x_0$ for all $x \in \{x_0\}$.

Example. (1) Let $X = \mathbb{R}^n$ and $A = B[0, 1] = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, the unit ball in $\mathbb{R}^n$. Then A is a retract of X.

Solution. Define $r : X \to A$ by

$$r(x) = \begin{cases} x & \text{if } \|x\| \leq 1, \\ \frac{x}{\|x\|} & \text{if } \|x\| > 1, \ \ x \in X. \end{cases}$$

Then r is continuous with $r(X) \subseteq A$ (since $||r(x)|| \leq 1$ for all $x \in X$) and $r(x) = x$ for all $x \in A$.([Dug66], p.322). (In particular, $[-1, 1]$ is a retract of $\mathbb{R}$).

Example. (2) $S^n(0, 1) = \{x \in \mathbb{R}^{n+1} : \|x\| = 1\}$ is a retract of $\mathbb{R}^{n+1}\backslash\{0\}$.

Note that, in particular, (i) $S^1(0, 1)$, the unit circle, is a retract of $\mathbb{R}^2\backslash\{0\}$;

(ii) $S^0(0, 1) = \{x \in \mathbb{R} : |x| = 1\} = \{+1, -1\}$ is a retract of $\mathbb{R}\backslash\{0\}$.

Theorem 1.1.10. ([Dug66], p. 322). *If X is Hausdorff, and $A \subseteq X$ a retract of X, then A is closed in X.*

Proof. Suppose $\overline{A} \neq A$ and let $x \in \overline{A}\backslash A$. Then $r(x) \neq x$. Since X is Hausdorff, $\exists$ open sets U & V such that $\{x\} \subseteq U$, $\{r(x)\} \subseteq V$ and $U \cap V = \varnothing$. Note that $r(U) \subseteq V$.

Since $x \in \overline{A}$ and U is a neighborhood of x, U must contain a point y of A. Since $r(y) = y$ and $r(U) \subseteq V$, we have $y \in V$. This implies that $y \in U \cap V$ and so $U \cap V \neq \varnothing$, a contradiction. □

Theorem 1.1.11. ([Dug66], p. 323). *Let X be a topological space and $A \subseteq X$. Then A is a retract of X iff each continuous map $f : A \to Y$ has a continuous extension $g : X \to Y$ for any topological space Y.*

Proof. ($\Longrightarrow$) Let $r : X \to A$ be a retract of X. Define $g : X \to Y$ by $g = f \circ r$. Clearly g is continuous on X. Further, g is an extension of $f : A \to Y$ since $g = f$ on A. In fact, for any $a \in A$, $g(a) = f(r(a)) = f(a)$.

($\Longleftarrow$) Take $Y = A$ and $f = i : A \to A = Y$. Then f is continuous and so, by hypothesis, $f : A \to Y$ has a continuous extension $g : X \to Y$. Then $r = g : X \to A$ is a retraction of X (since $r = g$ is continuous and, for any $a \in A$, $r(a) = g(a) = f(a) = i(a) = a$). □

Theorem 1.1.12. *Let $X = [a, b]$ is a closed subspace of $\mathbb{R}$, and let $A = \{a, b\}$. Then A is cannot be a retract of X.*

Proof. Let, if possible, $r : X \to A$ be a retract of X. Since X is connected and r is continuous, $r(X)$ is connected i.e $A = \{a, b\}$ is connected. But this is a contradiction, since $\{\{a\}, \{b\}\}$ is a disconnection (or separation) of A. $\Box$

Definition: Let X be a set or a topological space and $f : X \to X$ a function. Then a point $x \in X$ is called a **fixed point** of f if $f(x) = x$.

Theorem 1.1.13. (Fixed Point) *Every continuous function $f : [a, b] \to [a, b]$ has a fixed point.*

Proof. If $a = b$, clearly $f(a) = a$, so a is a fixed point of f. Now suppose $a < b$, and let $g : [a, b] \to [-1, 1]$ be a mapping defined by

$$g(x) = \frac{2x - (a + b)}{b - a}.$$

Then g is continuous and in fact a homeomorphism with $g(a) = -1$ and $g(b) = 1$. Hence we may assume that $a = -1$, $b = 1$ i.e $f : [-1, 1] \to [-1, 1]$ is a continuous mapping. Let if possible f has no fixed point. Then $f(x) \neq x$ for all $x \in [-1, 1]$.Define $r : [-1, 1] \to [-1, 1]$ by

$$r(x) = \frac{x - f(x)}{|x - f(x)|} \quad \text{for all } x \in [-1, 1].$$

Clearly (i) r is continuous,

(ii)

$$r(x) = \begin{cases} +1 \text{ if } x - f\ (x) > 0, \\ -1 \text{ if } x - f\ (x) < 0. \end{cases}$$

Therefore $r([-1, 1]) = \{-1, 1\}$. Hence $r : [-1, 1] \to \{-1, 1\}$ is a retraction, which contradicts the above theorem. Then $\exists$ at least one $x \in [-1, 1]$ such that $f(x) = x$. $\Box$

Lemma 1.1.14. (Intermediate Value Theorem) *Let $f : \mathbb{R} \to \mathbb{R}$ be a continuous function. Let $p, q \in f(\mathbb{R})$ with $p < q$. Then, for every $y \in R$ with $p < y < q$, there is an $x \in \mathbb{R}$ such that $f(x) = y$.* $\Box$

Proof. Since $\mathbb{R}$ is connected and $f : \mathbb{R} \to \mathbb{R}$ is continuous, $f(\mathbb{R})$ is connected. Let $p, q \in f(\mathbb{R})$ with $a < b$, and let $p < y < q$. Suppose there is no $x \in \mathbb{R}$ such that $f(x) = y$, i.e $f(x) \neq y$ for all $x \in \mathbb{R}$. If

$$A = f(\mathbb{R}) \cap (-\infty, y) \quad \& \quad B = f(\mathbb{R}) \cap (y, \infty),$$

then clearly $\{A, B\}$ is a disconnection of $f(\mathbb{R})$. This is a contradiction. Then $f(x) = y$ for some $x \in \mathbb{R}$. $\Box$

Theorem 1.1.15. (Fixed Point) *Let $I = [0, 1] \subseteq \mathbb{R}$. Then every continuous function $f : I \to I$ has a fixed point.*

Proof. Let $f : I \to I$ be a continuous function. If $f(0) = 0$ or $f(1) = 1$, then f has 0 or 1 as its fixed points. Suppose $f(0) \neq 0$, $f(1) \neq 1$. Then $f(0) > 0$ and $f(1) < 1$.Define a function $g : I \to \mathbb{R}$ by

$$g(x) = x - f(x) \text{ for all } x \in I.$$

Then clearly f is continuous. Further

$$g(0) = -f(0) < 0 \text{ and } g(1) = 1 - f(1) > 0$$

and so $g(0) < 0 < g(1)$. By the intermediate value theorem, there is a point $x \in I$ such that $g(x) = 0$. Then $x - f(x) = 0$ or $f(x) = x$ and so x is a fixed point of f. □

1.2 Normed and Topological Vector Spaces

In this section, we consider all vector spaces over the field $\mathbb{K}$ (= $\mathbb{R}$ or $\mathbb{C}$) with the usual absolute value.

Notations. Let E be a vector space and $A, B \subseteq E$, $x \in E$, α, $\beta \in \mathbb{K}$. We denote

$$\begin{aligned} x \pm A &= \{x \pm a : a \in A\}; \quad A \pm B = \{a \pm b : a \in A, b \in B\}; \\ \alpha A &= \{\alpha a : a \in A\}; \quad \alpha A + \beta B = \{\alpha a + \beta b : a \in A, b \in B\}. \end{aligned}$$

Note that $\alpha A + \beta A \subsetneqq (\alpha + \beta)A$, in general.

Definition: Let E be a vector space over $\mathbb{K}$. A function $p : E \to \mathbb{R}$ is called a **seminorm** on E if it satisfies

(i) $p(x) \geq 0$ for all $x \in E$;

(ii) $p(x) = 0$ if $x = 0$;

(iii) $p(\lambda x) = |\lambda| p(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ (*absolutely homogeneous*);

(iv) $p(x + y) \leq p(x) + p(y)$ for all $x, y \in E$ (*subadditive*).

Note. If p is a seminorm on a vector space E, it may happen that $p(x) = 0$ for some $x \neq 0$. If p satisfies $p(x) = 0$ implies $x = 0$, then p is a *norm* on E and we write $||.|| = p(.)$.

Definition: (1) A vector space E with a norm $\|.\|$ is called a **normed vector space** or, simply, a **normed space**. Clearly, every normed space $(E, \|.\|)$ is a metric space with metric given by:

$$d(x, y) = \|x - y\|, \quad x, y \in E.$$

(2) A normed space E is called a **Banach space** if it is complete with respect to the metric induced by the norm.

Definition: Let E be a vector space over the field $\mathbb{K}$ ($\mathbb{R}$ or $\mathbb{C}$) and τ a topology on E. Then the pair (E, τ) is called a **topological vector space (TVS)** if the following conditions hold:

(TVS$_1$) The operation of addition $(x, y) \to x+y$ of $E \times E \to E$ is jointly continuous; i.e., given any $x, y \in E$ and any τneighborhood U of $x+y$ in E, there exist τ-neighborhoods V of x and W of y in E such that $V + W \subseteq U$.

(TVS$_2$) The operation of scalar multiplication $(\lambda, x) \to \lambda x$ of $\mathbb{K} \times E \to E$ is jointly continuous; i.e., given any $x \in E$ and $\lambda \in \mathbb{K}$ and any τneighborhood U of λx in E, there exist a τneighborhood V of x in E and a neighborhood D of λ in $\mathbb{K}$ such that $DV \subseteq U$.

Examples. (1) Any normed space $(E, \|.\|)$ is a TVS since the conditions (TVS$_1$) and (TVS$_2$) follow, respectively, from:

(i) If $x, y \in E$ and $x_n \to x$, $y_n \to y$, then $x_n + y_n \to x + y$ in E.

(ii) If $x \in E$, $\lambda \in \mathbb{K}$ and $\lambda_n \to \lambda$, $x_n \to x$, then $\lambda_n x_n \to \lambda x$ in E.

(2) Any vector space E with the indiscrete topology $\tau_I = \{E, \varnothing\}$ is a TVS.

(3) A vector space E with the discrete topology $\tau_D = P(E)$ is not a TVS unless $E = \{0\}$ (since the scalar multiplication need not be continuous).

Definition: Let E be a vector space over the field $\mathbb{K}$ ($= \mathbb{R}$ or $\mathbb{C}$). A subset A of a vector space E is called:

(i) **absorbing** if, for each $x \in E$, there exists a number $r = r(x) > 0$ such that

$$x \in \lambda A \text{ for all } \lambda \in \mathbb{K} \text{ with } |\lambda| \geq r;$$

(ii) **balanced** if $\lambda x \in A$ for all $x \in A$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(iii) **convex** if $tx + (1-t)y \in A$ for all $x, y \in A$ and $0 \leq t \leq 1$.

Examples. (1) Every vector subspace A of E is balanced and convex.

(2) If $(E, ||.||)$ is a normed vector space, then, for each $r > 0$, both the open and closed balls

$$B(0, r) = \{x \in E : ||x|| < r\},\ B[0, r] = \{x \in E : ||x|| \leq r\}$$

are absorbing, balanced and convex subset of E.

Theorem 1.2.1. Every TVS (E, τ) *has a base of neighborhoods of* 0 *consisting of closed, balanced and absorbing sets.*

Definition: A TVS (E, τ) is called:

(i) a **locally convex space** (an **LCS**) if it has a base of neighborhoods of 0 consisting of convex sets (or equivalently its topology τ can be defined by a family of seminorms).

(ii) **metrizable** if τ is induced by a metric d on E.

Note. In general, E is locally convex $\not\Leftrightarrow$ E is metrizable.

Definition: Let E be a vector space over $\mathbb{K}$.

(1) A function $q : E \to \mathbb{R}$ is called an F**-norm** on E if it satisfies

(S_1) $q(x) \geq 0$ for all $x \in E$;

(S_2) $q(x) = 0$ iff $x = 0$;

(S_3) $q(\lambda x) \leq q(x)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

(S_4) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$ (*subadditive*).

(2) A function $q : E \to \mathbb{R}$ q is called a p**-norm** on E, where $0 < p \leq 1$, if it satisfies

(P_1) $q(x) \geq 0$ for all $x \in E$;

(P_2) $q(x) = 0$ iff $x = 0$;

(P_3) $q(\alpha x) = |\alpha|^p \, q(x)$ for all $x \in E$ and $\alpha \in \mathbb{K}$;

(P_4) $q(x + y) \leq q(x) + q(y)$ for all $x, y \in E$.

Theorem 1.2.2. (Metrization Theorem) *A TVS (E, τ) is metrizable $\Leftrightarrow$ it is Hausdorff and has a countable base of τ-neighborhoods of 0. In this case, τ may be defined by a metric d such that*

(i) $d(x + z, y + z) = d(x, y) = d(x - y, 0)$ *for all* $x, y, z \in E$,

(ii) $d(\lambda x, 0) \leq d(x, 0)$ for all $x \in E$ and $\lambda \in \mathbb{K}$ with $|\lambda| \leq 1$;

hence τ *is given by an F-norm q, where $q(x) = d(x, 0)$ for all $x \in E$.*

Theorem 1.2.3. *Let X and Y be normed spaces. Then, for a linear mapping $T : X \to Y$, TFAE:*

(a) T is is continuous on X

(b) T is is continuous at $0 \in X$

(c) T is bounded, i.e. there exists a constant $M > 0$ such that

$$||T(x)||_Y \leq M||x||_X \text{ for all } x \in X.$$

Definition: A mapping $T : X \to Y$ is said to have a **closed graph** if its graph

$$G(T) = \{(x, Tx) : x \in X\}$$

is a closed subset of $X \times Y$; or equivalently, if $x_\alpha \to x \in X$ and $T(x_\alpha) \to y \in Y$, then $y = T(x)$.

Note. If a mapping $T : X \to Y$ is continuous, then T has a closed graph. Its converse is:

Closed Graph Theorem. (Banach) *Let X and Y be Banach spaces. If a linear mapping $T : X \to Y$ has closed graph, then T is continuous has a closed graph.*

Theorem 1.2.4. *Let $(E, \|.\|)$ and $(F, \|.\|)$ be normed spaces and $CL(E, F)$ the vector space of all continuous linear mappings $T : E \to F$. Then $CL(E, F)$ is a normed* *space with respect to the pointwise addition and scalar multiplication and the operator norm:*

$$\|T\| = \sup\{\|T(x)\| : x \in E, \|x\| \leq 1\}, T \in CL(E, F).$$

If F is a Banach space, then $CL(E, F)$ is also a Banach space.

Definition: Let E be a TVS or a normed space. Then any linear map $f : E \to \mathbb{K}$ is called a *linear functional* on E. The set of all continuous linear functionals on E is also a vector space, called the *topological dual* of E and is denoted by E^*. Clearly, $E^* = CL(E, \mathbb{K})$.

Theorem 1.2.5. (Hahn-Banach) *Let $(E, ||\cdot||)$ be a normed space, M a vector subspace of E and f a continuous linear functional on M. Then there exists a $g \in E^*$ (called the extension of f from M to E) such that*

$$g = f \text{ on } M \text{ and } \|g\|_E = \|f\|_M = \sup\{|f(y)| : y \in M, \|y\| \leq 1\}.$$

Corollary 1.2.6. *Let $(E, ||\cdot||)$ be a normed space. Then, for any $x_0 \neq 0$ in E, there exists a $g \in E^*$ such that*

$$g(x_0) = ||x_0|| \text{ and } ||g|| = 1.$$

In particular, E^* separates points of E (i.e. for any $x \neq 0$ in E, there exist a $g \in E^*$ such that $g(x) \neq 0$).

1.3 Weak Topologies on Normed Spaces

Definition: Let $(E, ||\cdot||)$ be a normed space.

(i) For any $f \in E^*$, the mapping $p_f : E \to \mathbb{R}$ defined by

$$p_f(x) = |f(x)|, \; x \in E,$$

is a seminorm p_f on E. The collection $\{p_f : f \in E^*\}$ of seminorms defines a locally convex topology on E, called the **weak topology** and denoted by $w = w(E, E^*)$ or $\sigma(E, E^*)$.

(ii) Similarly, for any $x \in E$, the mapping $q_x : E^* \to \mathbb{R}$ defined by

$$q_x(f) = |f(x)|, \; f \in E^*.$$

is a seminorm q_x on E^*. Then the collection $\{q_x : x \in E\}$ of seminorms defines a locally convex topology on E^*, called the **weak*** ***topology*** on E^* and denoted by $w^* = w(E^*, E)$ or $\sigma(E^*, E)$.

Theorem 1.3.1. *(a) $w \leq ||\cdot||$ and (E, w) is Hausdorff.*

(b) $(E, w)^* = (E, ||\cdot||)^*$.

(c) (Mazur) For any convex subset M *of* E, M *is* $||\cdot||$*-closed iff* M *is* w*-closed; in particular,* $\overline{M}^w = \overline{M}^{||\cdot||}$.

(d) (Mackey) For any subset M *of* E, M *is* $||\cdot||$*-bounded iff* M *is* w*-bounded.*

(e) If (E, τ) *is finite-dimensional, then* $||\cdot|| = w$.

Definition: Let $(E, ||.||)$ be a normed space. A sequence $\{x_n\}$ in E is said to be **weakly convergent** to $x \in E$ if

$$f(x_n) \to f(x) \text{ for all } f \in E^*.$$

Theorem 1.3.2. *Let* $(E, ||\cdot||)$ *be a normed space. If* $x_n \to x$ *weakly in* E, *then* $\{x_n\}$ *is norm bounded in* E *and*

$$||x|| \leq \lim_n \inf ||x_n||.$$

Definition: ([Sim63], p. 233) (1) Let $(E, ||\cdot||)$ be an normed space. For any $x \in E$, we define a map $\widehat{x} : E^* \longrightarrow \mathbb{K}$ by

$$\widehat{x}(f) = f(x), \quad f \in E^*.$$

Then $|\widehat{x}(f)| = |f(x)| \leq ||f||.||x||$ for all $f \in E^*$, and so $\widehat{x} \in E^{**} = (E^*)^*$.

(2) W define the evaluation map $\pi : E \longrightarrow E^{**}$ by

$$\pi(x) = \widehat{x}, \quad x \in E.$$

Clearly, $\widehat{x} = 0$ implies that $x = 0$ and so π is one-one. Hence we may identify E with a subspace $\widehat{E} = \{\widehat{x} : x \in E\}$ of E^{**}, so $\widehat{E} \subseteq E^{**} \subseteq (E^*)^*$.

Theorem 1.3.3. ([Sim63], p. 233) *If* E *is a Banach space, then:*

(a) $||\widehat{x}|| = ||x||, x \in E$; *hence* π *is an isometry.*

(b) $\pi : E \longrightarrow \pi(E) = \widehat{E}$ *is a homeomorphism.*

Reflexive Normed Spaces

Definition: Let $(E, ||\cdot||)$ be an normed space. If $\pi : E \longrightarrow E^{**}$ is onto (i.e. if $\widehat{E} = E^{**}$), then E is called *reflexive.*

Note. If $(E, ||\cdot||)$ is a reflexive normed space, then, since $\pi : E \longrightarrow E^{**}$ is an onto isometry and $E^{**} = L(E^*, K)$ is complete, it follows that $(E, ||\cdot||)$ is complete and hence a Banach space.

Example. For $1 < p < \infty$, the Banach spaces ℓ_p and L_p are reflexive. [If $q > 1$ is such that $\frac{1}{p} + \frac{1}{q} = 1$, then

$$(\ell_p^*)^* \cong (\ell_q)^* \cong \ell_p \text{ and } (L_p^*)^* \cong (L_q)^* \cong L_p.$$

Theorem 1.3.4. (a) *A Banach space E is reflexive iff every closed bounded subset B of E is w-compact iff the closed unit ball $\{x \in E : ||x|| \leq 1\}$ of E is w-compact. Hence any bounded sequence $\{x_n\}$ in a reflexive Banach space has a w-convergent subsequence.*

(b) *If E is reflexive, then every $f \in E^*$ attains its maximum on the closed unit ball of E (i.e., there exists $x \in E$ with $||x|| = 1$ and $||f|| = |f(x)|$).*

Theorem 1.3.5. ([Sim63], p. 233) (Banach-Alaoglu theorem) *If $(E, ||\cdot||)$ is a normed space, then the closed unit ball $B_1^* = \{f \in E : ||f|| \leq 1\}$ of E^* is w^**-compact in E^*.

Theorem 1.3.6. [Sim63, p. 234] Let $(E, ||\cdot||)$ is a Banach space, and let $S^* = (B_1^*, w^*)$, a compact Hausdorff space. For any $x \in X$, define a continuous map $\widehat{x} : S^* \to K$ by

$$\widehat{x}(f) = f(x),\ \ f \in S^*.$$

Then the map $\pi : x \longrightarrow \widehat{x}$ is an isometric isomorphism of E onto a closed vector subspace of the Banach space $C(S^*)$.

Remark. By above theorem, any general Banach space $(E, ||\cdot||)$ can be considered as a closed vector subspace of the function space $(C(X), ||\cdot||_\infty)$, where X is a compact Hausdorff space.

1.4 Hausdorff Metric on $cb(X)$ and its Properties

Definition: Let A be a non-empty subset of a metric space (X, d). For any $x \in X$, we define the distance from x to A by

$$d(x, A) := \inf\{d(x, a) : a \in A\}.$$

Lemma 1.4.1. *Let A be a subset of a metric space (X, d). Then:*
(a) $d(x, A) \leq d(x, a)$ *for all $a \in A$.*
(b) $d(x, A) = 0 \Leftrightarrow x \in \overline{A}$.
(c) *For any $x, y \in X$,*

$$| \ d(x, A) - d(y, A) \ | \leq d(x, y).$$

Proof. (a) By definition,

$$d(x, A) = \inf_{a \in A} d(x, y) \leq d(x, a) \ \text{ for all } y \in A \tag{1}$$

(b)

$$\begin{aligned} x \quad &\in \quad \overline{A} \\ &\iff \text{For each } \varepsilon > 0, \text{ the open ball } B(x,\varepsilon) \text{ contains a point } a_\varepsilon \text{ (say) of } A \\ &\iff \text{For each } \varepsilon > 0, d(x,A) \le d(x,a_\varepsilon) < \varepsilon \iff d(x,A) = 0. \end{aligned}$$

(c) Let $x, y \in X$. Then, for any $a \in A$, using (1),

$$d(x,A) \le d(x,a) \le d(x,y) + d(y,a), \text{ or } d(x,A) - d(x,y) \le d(y,a).$$

Taking $\inf_{a\in A}$

$$\begin{aligned} d(x,A) - d(x,y) &\le \inf_{a\in A} d(y,a) = d(y,A), \\ \text{or} \quad d(x,A) - d(y,A) &\le d(x,y). \end{aligned} \tag{2}$$

Similarly, interchanging x and y

$$\begin{aligned} d(y,A) - d(x,A) &\le d(y,x) = d(x,y), \\ \text{or} \quad -[d(x,A) - d(y,A)] &\le d(x,y). \end{aligned} \tag{3}$$

By (2) and (3)

$$| d(x,A) - d(y,A) | \le d(x,y). \qquad \square$$

Definition: Let (X,d) be a metric space and 2^X the family of all non-empty subsets of X. For any subsets $A, B \in 2^X$, define

$$d(A,B) := \sup_{a\in A} d(a,B) = \sup_{a\in A} \inf_{b\in B} \{d(a,b)\}.$$

Notations. Let $b(X)$ (resp., $cb(X), k(X)$) denote the family of all non-empty bounded (resp., closed and bounded, compact) subsets of X. Clearly,

$$k(X) \subseteq cb(X) \subseteq b(X) \subseteq 2^X.$$

The following lemma shows that the function d is only a pseude-metric on $b(X)$.

Lemma 1.4.2. *Let (X,d) be a metric space. Then for any A, B, C in $b(X)$, we have the following:*

(i) $d(A,B) \ge 0$.

(ii) $d(A,B) = 0$ *iff* $A \subseteq \overline{B}$; *in particular,* $d(A,A) = 0$.

(iii) $d(A,B) \le d(A,C) + d(C,B)$.

Proof. (i) This is clear.

(ii) Assume $d(A,B) = \sup_{a\in A} d(a,B) = 0$. Then $d(a,B) = 0$ for all $a \in A$. But $d(a,B) = 0$ iff $a \in \overline{B}$. Hence $A \subseteq \overline{B}$. Conversely, if $A \subseteq \overline{B}$, then for any $a \in A$, $a \in \overline{B}$, and so we have $d(a,B) = 0$. Hence

$$d(A,B) = \sup_{a\in A} d(a,B) = \sup_{a\in A}\{0\} = 0.$$

In particular, since $A \subseteq \overline{A}$, $d(A,A) = 0$.

(iii) For any $a \in A, b \in B, c \in C$, clearly $d(a,b) \leq d(a,c) + d(c,b)$. Hence

$$\begin{aligned} \inf_{b\in B} d(a,b) &\leq \inf_{b\in B}\{d(a,c) + d(c,b)\} = d(a,c) + \inf_{b\in B} d(c,b), \\ \text{or,}\quad d(a,B) &\leq d(a,c) + d(c,B). \end{aligned}$$

By taking infimum over C in both sides, we get

$$\begin{aligned} d(a,B) &\leq \inf_{c\in C} d(a,c) + \inf_{c\in C} d(c,B) \\ &\leq d(a,C) + \sup_{c\in C} d(c,B) = d(a,C) + d(C,B). \end{aligned}$$

Next, taking supremum over A, we get

$$\begin{aligned} \sup_{a\in A} d(a,B) &\leq \sup_{a\in A} d(a,C) + d(C,B)\}, \\ \text{or}\quad d(A,B) &\leq d(A,C) + d(C,B). \qquad \square \end{aligned}$$

Remark. We should mention here that, in general, $d(A,B) \neq d(B,A)$, and so the function d does not define a metric on $b(X)$. [For example, let $X = \mathbb{R}$ with the usual metric $d(x,y) = |x-y|$, where $x, y \in \mathbb{R}$. Take $A = [1,2]$ and $B = \{3\}$. We have

$$\begin{aligned} d(A,B) &= \sup_{a\in A}\{\inf_{b\in B} |a-b|\} = \sup_{a\in[1,2]} |a-3| = |1-3| = 2, \\ d(B,A) &= \sup_{b\in B}\{\inf_{a\in A} |b-a|\} = \sup_{b=3}\{\inf_{a\in[1,2]} |3-a|\} = \sup_{b=3}\{|3-2|\} = 1.] \end{aligned}$$

Definition: Let (X,d) be a metric space. The *Hausdorff metric* on $cb(X)$, denoted by d_H, is the function $cb(X) \times cb(X) \to \mathbb{R}$ defined by

$$d_H(A,B) := \max\{d(A,B), d(B,A)\}, \quad A, B \in cb(X).$$

Theorem 1.4.3. *The function d_H is a metric on $cb(X)$.*

Proof. (M_1) Clearly $d_H(A,B) \geq 0$ for all $A, B \in cb(X)$.

(M_2) For $A, B \in cb(X)$, $d_H(A,B) = 0$ iff $d(A,B) = 0$ and $d(B,A) = 0$ iff $A \subseteq \overline{B} = B$ and $B \subseteq \overline{A} = A$; that is, $A = B$.

(M_3) For A, B in $cb(X)$, clearly $d_H(A,B) = d_H(B,A)$ for all $A, B \in cb(X)$,

(M_4) For $A, B, C \in cb(X)$,

$$d(A,B) \leq d(A,C) + d(C,B) \leq d_H(A,C) + d_H(C,B).$$

Similarly, $d(B,A) \leq d_H(B,C) + d_H(C,A)$. Combining these inequalities,

$$\begin{aligned} d_H(A,B) &= \max\{d(A,B),\ d(B,A)\} \\ &\leq \max\{d_H(A,C) + d_H(C,B),\ d_H(B,C) + d_H(C,B)\} \\ &= d_H(A,C) + d_H(C,B). \end{aligned}$$

Therefore, d_H is a metric on $cb(X)$. $\square$

Lemma 1.4.4. *If (X,d) is a metric space, then, for $A, B \in cb(X)$ and $x \in X$, we have*

$$|d(x,A) - d(x,B)| \leq d_H(A,B).$$

Proof. For any $y \in B$,

$$d(x,A) = \inf_{a\in A} d(x,a) \leq d(x,y) + \inf_{a\in A} d(y,a) \leq d(x,y) + d(B,A).$$

Taking $\inf_{y\in B}$,

$$\begin{aligned} d(x,A) &\leq \inf_{y\in B} d(x,y) + d(B,A) = d(x,B) + d(B,A), \\ \text{or } d(x,A) - d(x,B) &\leq d(B,A) \leq d_H(A,B). \end{aligned} \tag{1}$$

Exchange A and B to obtain that

$$\begin{aligned} d(x,B) - d(x,A) &\leq d(A,B) \leq d_H(A,B) \\ \text{or} \quad -[d(x,A) - d(x,B)] &\leq d_H(A,B). \end{aligned} \tag{2}$$

Thus, by (1) and (2),

$$|d(x,A) - d(x,B)| \leq d_H(A,B). \qquad \square$$

Definition: Let A be subset of a metric space (X,d). By ϕ_A, we mean the function $\phi_A : X \to \mathbb{R}$ defined by

$$\phi_A(x) = d(x,A) \quad \text{for all } x \in X.$$

By Lemma 1.4.1,

$$|\ \phi_A(x) - \phi_A(y)\ | = |d(x,A) - d(y,A)| \leq d(x,y) \quad \text{for all } x, y \in X.$$

Then, for any $\varepsilon > 0$,

$$|\ \phi_A(x) - \phi_A(y)\ | < \varepsilon \text{ whenever } x, y \in X \text{ with } d(x,y) < \varepsilon.$$

Hence the function ϕ_A is uniformly continuous on X and, in particular, continuous on X.

Theorem 1.4.5. *If a metric space (X, d) is complete , then $(cb(X), d_H)$ is also complete.*

Proof. Let $\{A_n\}$ be a Cauchy sequence in $(cb(X), d_H)$ and let $\varepsilon > 0$ be given, there is $N \geq 1$ such that

$$d_H(A_n, A_m) < \varepsilon \text{ for all } n, m \geq N.$$

For any $x \in X$ and any $n, m \geq N$, we have by above lemma,

$$| \phi_{A_n}(x) - \phi_{A_m}(y) |= |d(x, A_n) - d(x, A_m)| \leq d_H(A_n, A_m) < \varepsilon.$$

Therefore $\{\phi_{A_n}(x)\}$ is a Cauchy sequence in $\mathbb{R}$. Now each $\phi_{A_n} : X \to \mathbb{R}$ is continuous. Since X and hence $C(X, \mathbb{R})$ are complete, there is $g \in C(X, \mathbb{R})$ such that $\phi_{A_n} \to g$ in $C(X, \mathbb{R})$. Put $A_0 = \{x \in X : g(x) = 0\}$. Then $A_0 \in cb(X)$ and $\phi_{A_n} \overset{d_H}{\to} A_0$. Therefore, $(cb(X), d_H)$ is complete. □

Definition: Let X and Y be non-empty sets. Then a function $\varphi : X \to 2^Y$ is called **set-valued** *(or* ***multi-valued) mapping*** *or a* ***carrier*** if, for each $x \in X$, the set $\varphi(x)$ is not empty.

Example. If a (single-valued) function $f : X \to Y$ is onto, then, for each $y \in Y$, let

$$f^{-1}(y) = \{x \in X : f(x) = y\} \subseteq X.$$

Clearly the inverse map $f^{-1} : Y \to 2^X$ is a set-valued map.

We shall later study that if M is a "**proximinal**" subset of a metric space X, then the "**proximity map**" $P_M : X \to 2^M$ is set-valued.

1.5 Strictly and Uniformly Convex Metric Linear Spaces

Definition: A topological vector space (TVS) (E, τ) is called **metrizable** if its topology τ is induced by a metric d on E. Further, τ can also be given by an F-norm q, where $q(x) = d(x, 0)$ for all $x \in E$. In this case, we write $(E, \tau) = (E, q)$ and call it a **metric linear space**.

Definition: A complete metrizable TVS E is called an **F-space**.

For convenience, we may denote a metric linear space (X, d) by (X, q), where q is an F-norm on X.

Definition: [ANT77] A metric linear space (X, q) is called **strictly convex** if, any $r > 0$ and $x, y \in X$ with $q(x) \leq r$, $q(y) \leq r$,

$$x \neq y \Rightarrow q(\frac{x+y}{2}) < r.$$

Theorem 1.5.1. ([GK90], p. 34; [KK01], p. 138) *Let $(X, \|.\|)$ be normed space. Then the following conditions are equivalent:*

(a) *X is strictly convex.*

(b) *For any $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$,*

$$\begin{aligned} x &\neq y \Rightarrow \|x+y\| < 2, \\ \text{or equivalently, } \|x+y\| &= 2 \Rightarrow x = y. \end{aligned}$$

(c) *For any $x, y \in X$,*

$$\|x+y\| = \|x\| + \|y\| \text{ and } y \neq 0 \Longrightarrow x = ty \textit{ for some } t \geq 0.$$

(d) *For any $x, y, z \in X$,*

$$\|x-y\| = \|y-z\| = \frac{1}{2}\|x-y\| \Longrightarrow z = \frac{x+y}{2}.$$

Example (1). $(\mathbb{R}, |\cdot|)$ *is strictly convex, where $|\cdot|$ is the usual absolute value.*

Solution. Here $q(x) = |x|$, $x \in \mathbb{R}$. Let $x, y \in \mathbb{R}$ and $r > 0$ with $|x| \leq r$, $|y| \leq r$, $x \neq y$. We need to show that $|x+y| < 2r$.

Case I: Suppose $|x| < r$ or $|y| < r$, then

$$|x+y| \leq |x| + |y| < r + r = 2r.$$

Case II: Suppose $|x| = r$ and $|y| = r$. Since $x \neq y$, either $x = r$, $y = -r$, or $x = -r$, $y = r$. Hence in either case,

$$|x+y| \leq |0| = 0 < 2r. \quad \square$$

Alternatively: Let $x, y \in X$ with $|x| = 1$, $|y| = 1$. Suppose $x \neq y$. Then either $x = 1$, $y = -1$, or $x = -1$, $y = 1$. Hence in either case,

$$|x+y| = |0| = 0 < 2. \quad \square$$

Example (2). $(\mathbb{R}, q)$, *where*

$$q(x) = \frac{|x|}{1+|x|}, \ x \in \mathbb{R},$$

is strictly convex.

Solution. Let $x, y \in \mathbb{R}$ and $r > 0$ with $q(x) \leq r$, $q(y) \leq r$, and $x \neq y$. we show that $q(\frac{x+y}{2}) < r$. If $r = 1$

$$q\left(\frac{x+y}{2}\right) = \frac{|\frac{x+y}{2}|}{1+|\frac{x+y}{2}|} < 1 = r.$$

Suppose $r \neq 1$. Then

$$\begin{aligned} q(x) \quad &\leq \quad r \Longrightarrow \frac{|x|}{1+|x|} \leq r \Longrightarrow |x| \leq r + r|x| \\ &\Longrightarrow \quad |x|(1-r) \leq r \Longrightarrow |x| \leq \frac{r}{1-r}. \end{aligned}$$

Similarly $q(y) \leq r \;\; \Rightarrow \;\; |y| \leq \frac{r}{1-r}$. Since $(\mathbb{R}, |\cdot|)$ is strictly convex. by above example, $|\frac{x+y}{2}| < \frac{r}{1-r}$, hence

$$\left|\frac{x+y}{2}\right| - r\left|\frac{x+y}{2}\right| < r, \text{ or } \left|\frac{x+y}{2}\right| < r\left(1 + \left|\frac{x+y}{2}\right|\right);$$

$$\text{hence} \quad q\left(\frac{x+y}{2}\right) = \frac{|\frac{x+y}{2}|}{1+|\frac{x+y}{2}|} < r.$$

So $(\mathbb{R}, q)$ is strictly convex. $\square$

Theorem 1.5.2. Every inner product space $(X, <,>)$ is strictly convex.

Proof. Let $x \neq y \in X, \; \|x\| = 1, \; \|y\| = 1$. We show that $\|\frac{x+y}{2}\| < 1$. Applying parallelogram law:

$$\|x+y\|^2 + \|x-y\|^2 = 2\|x\|^2 + 2\|y\|^2,$$

$$\begin{aligned} \text{or,} \;\; \|x+y\|^2 \;\; &= \;\; 2\|x\|^2 + 2\|y\|^2 - \|x-y\|^2 \\ &= \;\; 2 + 2 - \|x-y\|^2 < 4 \quad \text{(since } x \neq y). \end{aligned}$$

Hence $\|x+y\| < 2$, and so $(X, <,>)$ is strictly convex. $\square$

Note. A normed space need not be strictly convex.

Example 1. Let $X = \mathbb{R}^2$ with the norm

$$\|x\|' = |x_1| + |x_2|, \; x = (x_1, x_2) \in \mathbb{R}^2.$$

Then $(\mathbb{R}^2, \|.\|')$ is not strictly convex.

Solution. Take $x = (1,0), \; y = (0,1)$, clearly $x \neq y$ and

$$\|x\|' = |1| + |0| = 1, \; \|y\|' = |0| + |1| = 1,$$

$$\left\|\frac{x+y}{2}\right\|' = \|(\frac{1}{2}, \frac{1}{2})\| = \frac{1}{2} + \frac{1}{2} = 1.$$

This shows that $(\mathbb{R}^2, \|.\|')$ is not strictly convex.

Example 2. Let $X = \mathbb{R}^2$ with the norm

$$\|x\|'' = \max\{|x_1|, |x_2|\}, \; x = (x_1, x_2) \in \mathbb{R}^2.$$

Then $(\mathbb{R}^2, \|.\|'')$ is not strictly convex.

Solution. Take $x = (1, 1),\ y = (-1, 1)$, clearly $x \neq y$ and

$$\|x\|'' = \max\{1, 1\} = 1,\ \|y\|'' = \max\{1, 1\} = 1,$$
$$\left\|\frac{x + y}{2}\right\|'' = \|(0, 1)\| = \max\{0, 1\} = 1.$$

This shows that $(\mathbb{R}^2, \|.\|'')$ is not strictly convex. $\square$

Example 3. Let $X = C[a, b]$ with the sup norm

$$\|f\| = \sup_{t\in[a,b]} |f(t)|, \quad f \in C[a, b].$$

Then $(C[a, b], \|.\|)$ is not strictly convex.

Solution. Consider $f, g \in C[a, b]$ given by

$$\begin{array}{ll} f(t) = 1 & \text{for all } t \in [a, b] \\ g(t) = \frac{t-a}{b-a} & \text{for all } t \in [a, b]. \end{array}$$

Then

$$\begin{aligned} \|f\| &= \sup_{t\in[a,b]} |f(t)| = \sup_{t\in[a,b]} 1 = 1 \\ \|g\| &= \sup_{t\in[a,b]} |g(t)| = \sup_{t\in[a,b]} \left|\tfrac{t-a}{b-a}\right| = \left|\tfrac{b-a}{b-a}\right| = 1. \end{aligned}$$

Clearly $f \neq g$. Now

$$\|f + g\| = \sup_{t\in[a,b]} |f(t) + g(x)| = \sup_{t\in[a,b]} (1 + \frac{t - a}{b - a}) = 1 + \frac{b - a}{b - a} = 2.$$

Hence we have $f, g \in [a, b]$ such that $\|f\| = 1,\ \|g\| = 1,\ f \neq g$, but $\|f + g\| = 2$. Therefore $(C[a, b], \|.\|)$ is not strictly convex. $\square$

Definition: [ANT77] A metric linear space (X, q) is called **uniformly convex** (UC) if, given any two number $r > 0,\ \varepsilon > 0$, there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$ with $q(x) \leq r,\ q(y) \leq r$

$$q(x - y) \geq \varepsilon \Longrightarrow q\left(\frac{x + y}{2}\right) < r - \delta. \tag{$*$}$$

Note that, by $(*)$, it is easy to guess or determine the value of δ.

Example 1. $(\mathbb{R}, |\cdot|)$ is uniformly convex, where $|\cdot|$ is the usual absolute value.

Solution. Let $X = \mathbb{R}$, and let $r > 0,\ \varepsilon > 0$. We show that there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in \mathbb{R}$ with $|x| \leq r,\ |y| \leq r,\ |x - y| \geq \varepsilon$, we have

$$|\frac{x + y}{2}| < r - \delta.$$

We take $\delta = \frac{\varepsilon}{4}$. Now, $-r \leq x \leq r$, $-r \leq y \leq r$, and $|x - y| \geq \varepsilon$.

Case I: Suppose $x - y \geq \varepsilon$. Then

$$x + y \leq x + (x - \varepsilon) = 2x - \varepsilon \leq 2r - \varepsilon < 2r - \frac{\varepsilon}{2} = 2(r - \frac{\varepsilon}{4}),$$
$$x + y \geq (y + \varepsilon) + y = 2y + \varepsilon \geq -2r + \varepsilon > -2r + \frac{\varepsilon}{2} = -2(r - \frac{\varepsilon}{4}).$$

Hence

$$-(r - \frac{\varepsilon}{4}) < \frac{x+y}{2} < r - \frac{\varepsilon}{4}, \text{ or } \left|\frac{x+y}{2}\right| < r - \frac{\varepsilon}{4} = r - \delta.$$

Case II: Suppose $x - y \leq -\varepsilon$. Then

$$x + y \leq (y - \varepsilon) + y = 2y - \varepsilon \leq 2r - \varepsilon < 2r - \frac{\varepsilon}{2} = 2(r - \frac{\varepsilon}{4}).$$
$$x + y \geq x + (x + \varepsilon) = 2x + \varepsilon \geq -2r + \varepsilon > -2r + \frac{\varepsilon}{2} = -2(r - \frac{\varepsilon}{4}).$$

Hence

$$-(r - \frac{\varepsilon}{4}) < \frac{x+y}{2} < r - \frac{\varepsilon}{4}, \text{ or } \left|\frac{x+y}{2}\right| < r - \frac{\varepsilon}{4} = r - \delta. \ \Box$$

Example 2. [ANT77] $(\mathbb{R}, q)$, where $q(x) = \frac{|x|}{1+|x|}$, $x \in \mathbb{R}$ is uniformly convex.

Solution. Let $X = \mathbb{R}$, and let $\varepsilon > 0$, $r > 0$ be given. We show that there exists $\delta = \delta(\varepsilon) > 0$ such that for any $x, y \in X$, $q(x) \leq r$, $q(y) \leq r$,

$$q(x - y) \geq \varepsilon \Rightarrow q(\frac{x+y}{2}) < r - \delta.$$

First consider $x, y \in X$ with $q(x) \leq r$, $q(y) \leq r$ and $q(x - y) \geq \varepsilon$. Then

$$\begin{aligned} \frac{|x|}{1+|x|} &= q(x) \leq r \Rightarrow |x| \leq r + r|x| \\ &\Rightarrow |x|(1-r) \leq r \Rightarrow |x| \leq \frac{r}{1-r} \end{aligned}$$

Similarly $q(y) \leq r$ gives $|y| \leq \frac{r}{1-r}$. Also $|x - y| > \frac{|x-y|}{1+|x-y|} = q(x-y) \geq \varepsilon$. Since $(\mathbb{R}, |\cdot|)$ is uniformly convex (by above example), so there exists $\delta_1 > 0$ such that

$$\left|\frac{x+y}{2}\right| < \frac{r}{1-r} - \delta_1 = \frac{r - \delta_1(1-r)}{1-r}. \tag{1}$$

Now

$$\begin{aligned} q\left(\frac{x+y}{2}\right) &= \frac{|\frac{x+y}{2}|}{1+|\frac{x+y}{2}|} < \frac{r - \delta_1(1-r)}{1 - r + r - \delta_1(1-r)} \\ &= r - [r - \frac{r - \delta_1(1-r)}{1 - \delta_1(1-r)}] = r - \frac{\delta_1(1-r)^2}{1 - \delta_1(1-r)}. \end{aligned}$$

Take

$$\delta = \frac{\delta_1(1-r)^2}{1 - \delta_1(1-r)} > 0 \quad (\text{since } \ 1 - \delta_1(1-r) > 0).$$

Then with this δ, we have

$$q\left(\frac{x+y}{2}\right) < r-\delta.$$

Hence $(\mathbb{R}, q)$ is uniformly convex. $\square$

Note. The following example shows that a finite-dimensional metric linear spaces need not be uniformly convex.

Example 3. The set $X = \mathbb{R}^2$ with metric

$$d(x,y) = \max\{|x_1-y_1|, |x_2-y_2|\},\ x=(x_1,x_2),\quad y=(y_1,y_2)\quad \in \mathbb{R}^2.$$

is not uniformly convex.

Solution. Here $q(x) = \max\{|x_1|, |x_2|\}$, $x = (x_1, x_2) \in \mathbb{R}^2$. Let $x = (1,1)$, $y = (1,0)$. Then

$$q(x) = \max\{1,1\} = 1,\ q(y) = \max\{1,0\} = 1$$
$$q(x-y) = q((0,1)) = \max\{0,1\} = 1.$$

So taking $r = 1$, $\varepsilon = 1$ we have for arbitrary $\delta = \delta(\varepsilon) > 0$, $q(x) = r$, $q(y) = r$, $q(x-y) = \varepsilon$ but,

$$q\left(\frac{x+y}{2}\right) = q\left(\frac{1+1}{2}, \frac{1+0}{2}\right) = q(1, \frac{1}{2}) = \max\{1, \frac{1}{2}\} = 1 = r.$$

Therefore $(\mathbb{R}^2, q)$ is not uniformly convex. $\square$

Theorem 1.5.3. *Every uniformly convex metric linear space (X, q) is strictly convex.*

Proof. Let $x, y \in X$ with $x \neq y$, $q(x) \leq r$, $q(y) \leq r$. We need to show that $q(\frac{x+y}{2}) < r$. [Since $x \neq y$, $q(x-y) > 0$. Take $\varepsilon = q(x-y) > 0$. Then, by uniformly convexity, there exists $\delta > 0$ such that

$$q(\frac{x+y}{2}) \leq r - \delta < r.]$$

Hence (X, q) is strictly convex. $\square$

Recall that:

(i) A subset K of a finite dimensional normed space $X \cong \mathbb{K}^n$ is compact iff K is closed and bounded.

(ii) Every compact subset A of a metric space (X, d) is both closed and bounded.

(iii) A subset A of a metric space (X, d) is compact iff it is both complete and totally bounded.

Definition: [ANT77] A metric linear space X is called **totally complete** if every closed and bounded subset of X is compact.

Examples. (1) Every finite dimensional normed space $(X, \|.\|)$ is totally complete. This follows from the fact that a subset S of $\mathbb{R}^n$ is compact iff S is closed and bounded.

(2). Every finite dimensional p-normed space $(X, \|.\|_p)$ is totally complete.

Theorem 1.5.4. [ANT77] *Any totally complete strictly convex metric linear space (X, q) is uniformly convex.*

Proof. Let $\varepsilon > 0,\ r > 0$ be given. Define

$$S = \{(x, y) \in X \times X : q(x) \leq r, q(y) \leq r, q(x - y) \geq \varepsilon\},$$

the metric on $X \times X$ being

$$d_1((x_1, y_1), (x_2, y_2)) = [\{q(x_1 - x_2)\}^2 + \{q(y_1 - y_2)\}^2]^{\frac{1}{2}}.$$

Then (i) S **is closed**: To show that $\overline{S} \subseteq S$, let $(x, y) \in \overline{S}$. Then there exists a sequence $\{(x_n, y_n)\}$ in S such that $(x_n, y_n) \to (x, y)$. Then $x_n \to x$ and $y_n \to y$. Also

$$q(x_n) \leq r,\ q(y_n) \leq r,\ q(x_n - y_n) \geq \varepsilon.$$

Letting $n \to \infty$ and by continuity of q, we have $q(x) \leq r$, $q(y) \leq r$, $q(x - y) \geq \varepsilon$, and so $(x, y) \in S$.

(ii) S **is bounded**: Clearly, by definition of S, $S \subseteq B[0, r] \times B[0, r]$; so S is bounded in $X \times X$.

Next, $(X \times X, d_1)$ is totally complete: Since (X, q) is a metric linear space, $(X \times X, d_1)$ is a metric linear space. Also if (X, q) is totally complete, then $(X \times X, d_1)$ is totally complete. Let K be a closed and bounded subset of $X \times X$. We show that K is compact. Since K is bounded, there exist $r > 0$ such that $K \subseteq B[0, r] \times B[0, r]$. But $B[0, r]$ is closed and bounded subset of X, and since X is totally complete, $B[0, r]$ is compact. $B[0, r] \times B[0, r]$ is compact in $X \times X$. Since K is closed subset of compact set $B[0, r] \times B[0, r]$, then K is compact. Therefore $X \times X$ is totally complete. S, being a closed and bounded subset of totally complete metric linear space is compact.

Define $\varphi : S \to \mathbb{R}$ by

$$\varphi(x, y) = r - q(\frac{x + y}{2}) \text{ for all } (x, y) \in S.$$

Then

$$\begin{aligned}
q(x) &\leq r,\ q(y) \leq r,\ q(x - y) \geq \varepsilon > 0 \\
&\Rightarrow q(x) \leq r,\ q(y) \leq r,\ x \neq y \\
&\Rightarrow q\left(\frac{x + y}{2}\right) < r \qquad \text{(by strict convexity of } q) \\
&\Rightarrow r - q\left(\frac{x + y}{2}\right) > 0 \qquad (1)
\end{aligned}$$

Now to show that $\varphi : X \times X \to \mathbb{R}$ is continuous on $X \times X$ (hence on S), suppose $\{(x_n, y_n)\}$ is a sequence in $X \times X$ such that $(x_n, y_n) \to (x, y)$ in $X \times X$. Then $x_n \to x$ in X and $y_n \to y$ in X and so by continuity of q

$$q\left(\frac{x_n + y_n}{2}\right) \to q\left(\frac{x+y}{2}\right) \qquad \text{in } \mathbb{R};$$
$$\text{hence } \ \varphi(x_n, y_n) = r - q\left(\frac{x_n + y_n}{2}\right) \to r - q\left(\frac{x+y}{2}\right) = \varphi(x, y).$$

Let $\delta = \inf\{\varphi(x, y) : (x, y) \in S\}$. Since φ is continuous on compact set S, φ will attain its infimum on S. So there exist $(x', y') \in S$ such that $\delta = \inf \varphi(x, y) = \varphi(x', y') > 0$ by (1). Now, for all $(x, y) \in S$,

$$\delta \leq \varphi(x, y) = r - q\left(\frac{x+y}{2}\right), \text{ or } \ q\left(\frac{x+y}{2}\right) \leq r - \delta.$$

Hence (X, q) is uniformly convex. □

1.6 Convexity of Open Balls in Metric Linear Spaces

In this section we present some results of T.S. Norfolk [Nor] and K. Sastry and S. Naidu [SN79].

Let (E, d) be a metric space. One natural question which arises is the following: What metric spaces have the property that all open balls in the space are convex ? More precisely, we consider:

Question 1: If E is a linear space and d is a metric on E, is the open ball $B(x, r)$ is convex ?

In general, this is indeed false. To see this, we have:

Counter-example: ([Nor], p. 2-3) Consider the linear space $E = \mathbb{R}^2$, with the metric

$$d(x, y) = d((x_1, x_2), (y_1, y_2)) = \sqrt{|x_1 - y_1|} + \sqrt{|x_2 - y_2|}.$$

Then $B(x, r)$ is not convex for any $x \in E$, $r > 0$.

Solution. Clearly, d is metric on E, which can be proved using the fact that

$$\sqrt{a+b} \leq \sqrt{a} + \sqrt{b} \text{ for } a, b \geq 0.$$

Given any $x = (x_1, x_2) \in E$ and $r > 0$, choose $\frac{r^2}{2} < \alpha < r^2$, and let $y = (x_1 + \alpha, x_2)$, $z = (x_1, x_2 + \alpha)$.Then,

$$\begin{aligned} d(x, y) &= \sqrt{|x_1 + \alpha - x_1|} + \sqrt{|x_2 - x_2|} = \sqrt{\alpha} < \sqrt{r^2} = r, \\ \text{similarly } d(x, z) &< r \end{aligned}$$

so that $y, z \in B(x, r)$. However,

$$\begin{aligned} d(x, \frac{y+z}{2}) &= d((x_1, x_2), (x_1 + \frac{\alpha}{2}, x_2 + \frac{\alpha}{2}) \\ &= \sqrt{\frac{\alpha}{2}} + \sqrt{\frac{\alpha}{2}} = 2\sqrt{\frac{\alpha}{2}} = \sqrt{2\alpha} > \sqrt{r^2} = r, \end{aligned}$$

so that $\frac{y+z}{2} \notin B(x, r)$. Therefore $B(x, r)$ is not convex. □

Given that our previous example shows that not all metrics generate convex open balls, we are in a position to completely characterize those metrics which do so.

Theorem 1.6.1. ([Nor], p. 2-3) *Let (E, d) be a metric linear space.Then the open balls in X are convex iff*

$$d(x, ty + (1-t)z) \leq \max\{d(x, y), d(x, z)\},$$

for all $x, y, z \in E$ and $0 \leq t \leq 1$.

Proof. On one hand, suppose that the indicated condition holds. That is, given any $x \in E$, with $y, z \in B(x, r)$ and $0 \leq t \leq 1$, we have

$$d(x, ty + (1-t)z) \leq \max\{d(x, y), d(x, z)\} < r,$$

so that $ty + (1-t)z \in B(x, r)$, the desired convexity condition.

On the other hand, if there exist $x, y, z \in E$ and $0 \leq t \leq 1$ such that

$$d(x, ty + (1-t)z) > \max\{d(x, y), d(x, z)\},$$

we may choose

$$\max\{d(x, y), d(x, z)\} < r < d(x, ty + (1-t)z),$$

from which $y, z \in B(x, r)$, but $ty + (1-t)z \notin B(x, r)$, meaning that the latter set is not convex. □

Corollary 1.6.2. ([Nor], p. 5-6) *If any of the following conditions holds on a metric d on a linear space E, then Question 1 holds (i.e. the open balls of the metric space are convex).*

(1) *d is the discrete metric.*

(2) *For each fixed $x \in E$, d is a convex function on E. That is, for all $y, z \in E$ and $0 \leq t \leq 1$:*

$$d(x, ty + (1-t)z) \leq td(x, y) + (1-t)d(x, z)$$

(3) *d satisfies the following conditions :*

(a) *$d(x + y, x + z) = d(y, z)$ for all $x, y, z \in E$.*

(b) *$d(tx, ty) \leq td(x, y)$ for all $x, y \in E$ and $0 \leq t \leq 1$.*

(4) *There exists a norm $||.||$ on E such that $d(x, y) = ||x - y||$ for $x, y \in E$.*

Proof. (1) Suppose that d is the discrete metric. Then, for any $x \in E$, clearly

$$\begin{aligned} B(x,r) &= \{x\} \text{ if } 0 \le r \le 1 \\ \text{and } B(x,r) &= E \text{ if } r > 1. \end{aligned}$$

In either case, $B(x,r)$ is clearly convex.

(2) Suppose that the indicated condition holds. Then, for any $x, y, z \in E$ and $0 \le t \le 1$, we have

$$d(x, ty + (1-t)z) \le td(x,y) + (1-t)d(x,z) \le \max\{d(x,y), d(x,z)\},$$

which, by Theorem 1.6.1 means that all open balls are convex.

(3) Suppose that the given pair of conditions holds. We will show that this implies the conditions of (2). For any $x, y, z \in E$ and $0 \le t \le 1$, we have

$$\begin{aligned} d(x, ty + (1-t)z) &= d(x - x, ty + (1-t)z - x) \text{ (by a)} \\ &= d(x - x, ty + (1-t)z - x + [tx - tx]) \\ &= d(0, t(y-x) + (1-t)(z-x)) \\ &\le d(0, t(y-x)) + d(t(y-x), t(y-x) + (1-t)(z-x)) \\ &\le td(0, y-x) + d(0, (1-t)(z-x)) \text{ (by b)} \\ &\le td(x,y) + (1-t)d(0, z-x) \text{ (by a and b)} \\ &= td(x,y) + (1-t)d(x,z) \text{ (by a)}, \end{aligned}$$

the desired result.

(4) Suppose that d is derived from a norm. Then, it clearly satisfies the conditions of (3), since it is translation invariant, and also satisfies the stronger condition $d(tx, ty) = td(x,y)$ for all $x, y \in E$ and $0 \le t \le 1$. $\square$

Theorem 1.6.3. ([Nor], p. 7) *If d is a metric on the linear space E such that all open balls are convex, then the induced linear topological space is locally convex.*

On the other hand, if the metric d induces a locally convex topology on the linear space E, it is not necessarily true that all open balls are convex. In fact, it can be the case that no open balls in a locally convex metric space are themselves convex. To see this, consider the following:

Example 2. Let $E = \mathbb{R}^2$, and d be the metric of previous example. Then, none of the open balls is convex, but the metric space is locally convex.

We next consider the convexity of closed sets in a TVS.

Definition: Let X be a topological space and S be a subset of X. The *boundary* of S is defined as

$$\partial S := \overline{S} \cap \overline{(X - S)}.$$

For any $x, y \in X$, $x \neq y$, we write

$$(x, y) = \{tx + (1-t)y : 0 < t < 1\}.$$

From the following theorem it follows that a strictly convex metric linear space is locally convex.

Theorem 1.6.4. [SN79] *Let (X, τ) be a topological vector space and S a non-empty closed subset of X such that $x, y \in \partial S$ and $x \neq y$ imply $(x, y) \cap S \neq \emptyset$. Then S is convex.*

Proof. Suppose S is not convex. Then there exist $x, y \in S$, $x \neq y$ such that $(x, y) \cap (X \backslash S) \neq \emptyset$. Let

$$A = \{t \in (0, 1) : tx + (1-t)y \in X \backslash S\}.$$

Clearly, A is non-empty. Next, A is open, as follows. [Let $t_0 \in A$. Then

$$t_0 x + (1 - t_0)y \in X \backslash S.$$

Since $X \backslash S$ is open, there exists an open set U in X such that

$$t_0 x + (1 - t_0)y \in U \subseteq X \backslash S.$$

Define $\varphi : (0, 1) \to X$ by

$$\varphi(t) = tx + (1-t)y, \quad t \in (0, 1).$$

Since φ is continuous, there exists an open interval J contains t_0 such that $\varphi(J) \subseteq U \subseteq X \backslash S$, and so $t_0 \in J \subseteq A$. Hence A is an open subset of $\mathbb{R}$.] Let B be a component of A (i.e, B is maximal connected subset of A). Then, there exist $\alpha, \beta \in \mathbb{R}$ such that $\alpha < \beta$ and $B = (\alpha, \beta)$. Write $z_1 = \alpha x + (1-\alpha)y$ and $z_2 = \beta x + (1-\beta)y$. Clearly z_1, z_2 are distinct points of ∂S and $(z_1, z_2) \cap S = \emptyset$ which contradicts the hypothesis. □

Corollary 1.6.5. [SN79] *Let (X, d) be a strictly convex metric linear space. Then the balls in X are convex.*

Proof. Let $B[0, r]$ be a closed ball. For any $x, y \in \partial(B[0, r])$ such that $x \neq y$, we have $d(x, 0) = r$ and $d(y, 0) = r$. Since X is strictly convex, $d(\frac{x+y}{2}, 0) < r$. Clearly $\frac{x+y}{2} \in (x, y) \cap B[0, r]$, and so $(x, y) \cap B[0, r] \neq \emptyset$. So, by above Theorem, the closed ball $B[0, r]$ is convex. Hence the open balls with center at 0 are also convex. Now, any balls $B[x, r]$, $B(x, r)$ can be expressed as translates of convex balls $B[0, r]$, $B(0, r)$:

$$B[x, r] = x + B[0, r], \quad B(x, r) = x + B(0, r).$$

This implies that both $B[x, r]$ and $B(x, r)$ are convex. □

1.7 Best Approximations in Metric Linear Spaces

We recall that if $M \subseteq (X, d)$ and $x \in X$, the distance from x to M is defined by

$$d(x, M) := \inf\{d(x, m) : m \in M\}$$

Then: (a) $d(x, M) \leq d(x, m)$ for all $m \in M$.
(b) $d(x, M) = 0 \Leftrightarrow x \in \overline{M}$.
(c) For any $x, y \in X$,

$$| \, d(x, M) - d(y, M) \, | \leq d(x, y).$$

Definition: Let M be subset of a metric space (X, d). By ϕ_M, we mean the function $\phi_M : X \to \mathbb{R}$ defined by

$$\phi_M(x) = d(x, M) \text{ for all } x \in X.$$

Lemma 1.7.1. *The function $\phi_M : X \to \mathbb{R}$ defined above is uniformly continuous (hence continuous) on X*

Proof. Clearly,

$$| \, \phi_M(x) - \phi_M(y) \, | = |d(x, M) - d(y, M)| \leq d(x, y) \text{ for all } x, y \in X.$$

Then, for any $\varepsilon > 0$, take $\delta = \varepsilon$, so that

$$| \, \phi_M(x) - \phi_M(y) \, | < \varepsilon \text{ whenever } x, y \in X \text{ with } d(x, y) < \varepsilon.$$

Hence the function ϕ_M is uniformly continuous on X and, in particular, continuous on X.

Definition: Let M be a subset of a metric space (X, d). For each $x \in X$, let

$$P_M(x) := \{z \in M \, : \, d(x, z) = d(x, M)\}.$$

Then each $z \in M$ is called a point of best approximation of M from x (or nearest point of M from x).

Clearly $P_M(x) \subseteq M$. Hence P_M is a set-valued map from X into $p(M)$, the power set of M. Problems in Best Approximation Theory deal first to determine whether, for each $x \in X$, $P_M(x)$ is non-empty, or a singleton or empty. This depends on the space X and also on the subset M.

By the following Lemma, the interesting case to find $P_M(x)$ is that $x \in X \setminus \overline{M}$.

Lemma 1.7.2. *Let M be a subset of a metric space (X, d) and $x \in X$. Then*

$$P_M(x) = \begin{cases} \{x\} & \text{if} \quad x \in M \\ \varnothing & \text{if} \quad x \in \overline{M} \setminus M. \end{cases}$$

Proof. Case I: Let $x \in M$. Then

$$\begin{aligned} P_M(x) &= \{z \in M \,:\, d(x,z) = d(x,M) = 0\} \\ &= \{z \in M \,:\, d(x,z) = 0\} \\ &= \{z \in M \,:\, x = z\} = \{x\}. \end{aligned}$$

Case II: Let $x \in \overline{M} \setminus M$. Then

$$\begin{aligned} P_M(x) &= \{z \in M \,:\, d(x,z) = d(x,M) = 0\} \\ &= \{z \in M \,:\, d(x,z) = 0\} \\ &= \{z \in M \,:\, x = z\} = \varnothing. \qquad \square \end{aligned}$$

Note. 1. If $M \subseteq (X, \|\,.\,\|)$ a normed space, then $d(x,M) = \inf\{\| x - y \| : y \in M\}$.

2. If M is closed, then $M = \overline{M}$, so $d(x,M) = 0 \Leftrightarrow x \in M$.

3. By above lemma, the interesting case to find $P_M(x)$ is that $x \in X \setminus \overline{M}$. (or $x \in X \setminus M$ if M is closed).

Definition: Let M be a subset of a metric space (X, d). Then

1. M is called a **proximinal set** (or an existence set,or an E-set) if $P_M(x) \neq \varnothing$ for each $x \in X$.

2. M is called a **Chebyshev set** (or a unique existence set or UE-set) if $P_M(x)$ is a singleton set for each $x \in X$.

3. M is called a **semi-Chebyshev set** (or a **uniqueness set**,or an U-set) if $P_M(x)$ contains at most one element for each $x \in X$.

Example 1. *Let* $X = \mathbb{R}^2$ with metric

$$d(x,y) = \sqrt{(x_1 - y_1)^2 + (x_2 - y_2)^2},\ \ x = (x_1, x_2),\ \ y = (y_1, y_2)\ \ \in \mathbb{R}^2,$$

and let $M = x$-axis $= \{(\alpha, 0) : \alpha \in \mathbb{R}\}$. Then M is Chebyshev and hence proximinal.

Solution. For any $x = (p, q) \in X$,

$$\begin{aligned} d(x,M) &= \inf_{m \in M} d(x,m) \\ &= \inf_{\alpha \in \mathbb{R}} \sqrt{(p - \alpha)^2 + (q - 0)^2} = q. \end{aligned}$$

Next,

$$\begin{aligned} P_M(x) &= \{z \in M : d(x,z) = q\} \\ &= \{(\alpha, 0) \in M : \sqrt{(p-\alpha)^2 + q^2} = q\} \\ &= \{(\alpha, 0) \in M : |p - \alpha| = 0\} \\ &= \{(\alpha, 0) \in M : \alpha = p\} \\ &= \{(p, 0)\}. \end{aligned}$$

Thus M is Chebyshev, hence proximinal. $\square$

Example 2. *Let* $X = \mathbb{R}^2$*, with metric*

$$d'(x, y) = \max\{|x_1 - y_1|, |x_2 - y_2|\},\ x = (x_1, x_2),\ \ y = (y_1, y_2)\ \ \in \mathbb{R},$$

and let $M = x$*-axis*$= \{(\alpha, 0) : \alpha \in \mathbb{R}\}$*. Then* M *is proximinal, but not Chebyshev.*

Solution. For any $x = (p, q) \in X$,

$$\begin{aligned} d(x, M) &= \inf_{m\in M} d'(x, m) \\ &= \inf_{\alpha\in\mathbb{R}} \max\{|p - \alpha|, |q - 0|\} = |q|. \end{aligned}$$

Further,

$$\begin{aligned} P_M(x) &= \{z \in M : d'(x, z) = |q|\} \\ &= \{(\alpha, 0) \in M : \max\{|p - \alpha|, |q|\} = |q|\} \\ &= \{(\alpha, 0) \in M : |p - \alpha| \le |q|\} \\ &= \{(\alpha, 0) \in M : -|q| \le \alpha - p \le |q|\} \\ &= \{(\alpha, 0) \in M : p - |q| \le \alpha \le p + |q|\} \\ &= \text{the line segment joining } (p - |q|,0) \text{ and } (p + |q|,0). \end{aligned}$$

Clearly, $P_M(x)$ is non-empty, but not necessarily a singleton. Thus M is proximinal, but not Chebyshev. □

Example 3. Let $X = \mathbb{R}$ with the usual metric $d(x, y) = |x-y|$. Take $M = (-1, 1)$, an open interval. Let $x = 2$. Then

$$d(x, M) = d(2, M) = \inf_{m\in M} |2 - m| = 1.$$

Hence,

$$\begin{aligned} P_M(x) &= \{z \in M : d(x, z) = d(x, M) = 1\} \\ &= \{z \in (-1, 1) : |2 - z| = 1\} \\ &= \varnothing \qquad \text{(since } 1 \notin (-1, 1) = M). \end{aligned}$$

Therefore M is not proximinal, hence not Chebyshev; however, M is semi-Chebyshev.

Definition: Let K be a subset of a metric space (X, d). Then K is called a *bounded set* if there exists $c > 0$ such that $d(x, y) \le c$ for all $x, y \in K$ (i.e., if its diameter $\delta(K) = \sup_{x,y\in K} d(x, y) < \infty$).

Theorem 1.7.3. *Let* M *be a subset of a metric space* (X, d)*. Then for each* $x \in X$*,* $P_M(x)$ *is bounded and closed subset of* X*.*

Proof. $P_M(x)$ is a bounded: Let $z_1, z_2 \in P_M(x)$. If $d(x, M) = \gamma$, then $d(x, z_1) = d(x, z_2) = \gamma$. Now

$$\begin{aligned} d(z_1, z_2) &\le d(x, z_1) + d(x, z_2) \\ &\le \gamma + \gamma = 2\gamma \quad \text{(constant)}, \end{aligned}$$

since $z_1, z_2 \in P_M(x)$ are arbitrary, $\sup d(z_1, z_2) \leq 2\gamma$. Hence $P_M(x)$ is a bounded set.

$P_M(x)$ is a closed: To show that $\overline{P_M(x)} \subseteq P_M(x)$, let $z \in \overline{P_M(x)}$. Then there exist a sequence $\{z_n\} \subseteq P_M(x)$ such that $z_n \to z$. Clearly,

$$d(x, z_n) = d(x, M) = \gamma \quad \text{for all } n \geq 1 \qquad (\text{since } z_n \in P_M(x)).$$

Now, using continuity of d,

$$d(x, z) = d(x, \lim_{n\to\infty} z_n) = \lim_{n\to\infty} d(x, z_n) = \lim_{n\to\infty} \gamma = \gamma,$$

so $z \in P_M(x)$. Hence $\overline{P_M(x)} \subseteq P_M(x)$. Thus $P_M(x)$ is a closed set. $\square$

Theorem 1.7.4. *Let M be a proximinal subset of a metric space X. Then M is closed.*

Proof. Suppose M is not closed. Then there exist a sequence $\{z_n\}$ in M such that $z_n \to x$ and $x \notin M$. Then $d(x, M) \leq d(x, z_n) \to 0$, so that $d(x, M) = 0$. But $d(z, x) > 0$ for each $z \in M$, since $x \notin M$. Therefore

$$P_M(x) = \{z \in M : d(x, z) = d(x, M) = 0\} = \varnothing.$$

This contradicts $P_M(x) \neq \varnothing$. [**Another Proof.** Let $x \in \overline{M}$. Then $d(x, M) = 0$. Since $P_M(x) \neq \varnothing$,

$$P_M(x) = \{z \in M : d(x, z) = d(x, M) = 0\} = \{x\}.$$

So $x \in P_M(x) \subseteq M$.] $\square$

Definition: (1) Let X be a vector space over $\mathbb{K}$. For any $x, y \in X$, we define a set

$$[x, y] = \{(1-t)tx + ty : t \in \mathbb{R}, 0 \leq t \leq 1\},$$

called the **line segment** joining x and y.

(2) Let X be a vector space over $\mathbb{K}$ and $A \subseteq X$. Then A is **convex** if $[x, y] \subseteq A$ for all $x, y \in A$.

Examples. (1) For any vector space X , the sets $\varnothing$, X, and singletons $\{x\}$ are convex.

(2) Both the open and closed ball in a normed space X are convex.

(3) The intersection of any collection of convex sets in a vector space X is convex.

Theorem 1.7.5. *Let X be a normed space. If M is a convex subset of X, then $P_M(x)$ is also convex for each $x \in X$.*

Proof. Let $x \in X$, suppose that z_1 and z_2 are in $P_M(x)$ and $0 \leq t \leq 1$. If

$y = \lambda z_1 + (1-t)z_2$, then $y \in M$ and

$$\begin{aligned} d(x,M) &\leq \|y-x\| \\ &= \|t(z_1-x)+(1-t)(z_2-x)\| \\ &\leq \|t(z_1-x)\| + \|(1-t)(z_2-x)\| \\ &\leq t\|z_1-x\|+(1-t)\|z_2-x\| \\ &= td(x,M)+(1-t)d(x,M) = d(x,M); \\ \text{or} \quad \gamma &\leq \parallel y-x \parallel \leq \gamma \text{ or } \parallel y-x \parallel = \gamma, \end{aligned}$$

Hence $y = tz_1 + (1-t)z_2 \in P_M(x)$. Therefore $P_M(x)$ is convex. □

Recall that: If K is subset of a metric space (X,d), then K is compact iff every sequence $\{x_n\}_{n=1}^{\infty}$ in K has a convergent subsequence with limit in K.

Theorem 1.7.6. *Any compact subset M of a metric space (X,d) is proximinal.*

Proof. Let $x \in X$ and let $\gamma = d(x,M)$. Since $\gamma = \inf\{d(x,y) : y \in M\}$ so there exist a sequence $\{d(x,y_m) : y_m \in M\}$ such that

$$\lim_{m\to\infty} d(x,y_m) = \gamma. \tag{1}$$

Here $\{y_m\}$ is called a minimizing sequence in M. Since M is compact, $\{y_m\}_{m=1}^{\infty}$ has a convergent subsequence $\{y_{m_i}\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} y_{m_i} = z \in M$. Now by continuity of metric d,

$$\lim_{i\to\infty} d(x,y_{m_i}) = d(x, \lim_{i\to\infty} y_{m_i}) = d(x,z) \tag{2}$$

Then by (1) and (2) the sequence $\{d(x,y_m)\}$ and its subsequence $\{d(x,y_{m_i})\}$ both converge in $\mathbb{R}$, so they must have the same limit. Hence $d(x,z) = \gamma$. Therefore $z \in P_M(z)$, and so $P_M(x) \neq \varnothing$. Thus M is proximinal. □

Remarks. (1) If $\{y_m\}_{m=1}^{\infty}$ is a Cauchy sequence in a metric space (X,d) and if $\{y_m\}_{m=1}^{\infty}$ has a convergent subsequence $\{y_{m_i}\}_{i=1}^{\infty}$ with $\lim_{i\to\infty} y_{m_i} = y \ \in X$, then it follows that $\{y_m\}_{m=1}^{\infty}$ is convergent and $\lim_{i\to\infty} y_m = y$.

(2) If the sequence $\{y_m\}_{m=1}^{\infty}$ is not Cauchy, then the above need not be true: e.g., the sequence $\{y_m\}_{m=1}^{\infty} = \{(-1)^m\}$ in $\mathbb{R}$ has convergent subsequence $\{1,1,\cdots\}$ and also $\{-1,-1,\cdots\}$, but $\{y_m\}$ is not convergent.

Theorem 1.7.7. *Any finite dimensional vector subspace M of a normed space $(X,\|.\|)$ is proximinal.*

Proof. let $x \in X$, and $\gamma = d(x,M)$, we show that $P_M(x) \neq \varnothing$. there exist a minimizing sequence $\{y_m\} \subseteq M$ such that

$$\|x-y_m\| \to \gamma \tag{1}$$

we now show that $\{y_m\}$ is a bounded sequence. Since $\{\|x-y_m\|\}$ is a convergent sequence in $\mathbb{R}$ and hence bounded, there exist a constant $c>0$ such that

$$\|x-y_m\| \leq c \text{ for all } m \geq 1.$$

Then, for any $m \geq 1$

$$\|y_m\| = \|y_m - x + x\| \leq \|y_m - x\| + \|x\| \leq c + ||x|| \text{ (constant)}.$$

Hence $\{y_m\}$ is bounded in M.

Since M is finite dimensional, it has the Bolzano-Weierstrass property (i.e every bounded sequence in M has a convergent subsequence), so $\{y_m\}$ has a convergent subsequence $\{y_{m_i}\}$ with $\lim_{i\to\infty} y_{m_i} = z \ \in X$. Since M, being a finite dimensional vector subspace of X, is closed, so $\lim_{i\to\infty} y_{m_i} = z \ \in M$. Now

$$\|x-z\| = \|x - \lim_{i\to\infty} y_{m_i}\| = \lim_{i\to\infty} \|x - y_{m_i}\|. \tag{2}$$

Then by (1) and (2) the sequence $\{\|x-y_m\|\}$ and its subsequence $\{\|x-y_{m_i}\|\}$ both converge, so they must have the same limit. Hence, $\|x-z\| = \gamma$. Therefore $z \in P_M(x)$, and so $P_M(x) \neq \varnothing$. Thus M is proximinal. □

Theorem 1.7.8. [ANT77] *Every proximinal convex subset M of a strictly convex metric linear space (X,q) is Chebyshev.*

Proof. Let $x \in X \setminus M$. Since M is proximinal, it is closed and $P_M(x) \neq \varnothing$. So $\gamma = d(x,M) > 0$. To show that $P_M(x)$ is singleton, let $z, z' \in P_M(x)$. Then $q(x-z) = \gamma,\ q(x-z') = \gamma$. Since M is convex, $\frac{z+z'}{2} \in M$. Now

$$\gamma = d(x,M) \leq q(x - \frac{z+z'}{2}) = q(\frac{(x-z)+(x-z')}{2}).$$

By strict convexity of X, $x-z = x-z'$, or $z=z'$. Thus M is Chebyshev. □

Corollary 1.7.9. [ANT77] *Any non-empty convex subset of strictly convex metric linear space is semi-Chebyshev.*

Definition: Let M be a subset of any metric space X. Then M is called approximatively compact, if for any $x \in X$ and any sequence $\{x_n\}$ in M, the condition $\lim_{n\to\infty} d(x,x_n) = d(x,M)$ implies that $\{x_n\}$ has a subsequence $\{x_{n_k}\}$ convergent to a point in M.

Recall that if M is any subset of a metric space (X,d), then M is compact iff every sequence $\{x_n\}$ in M has convergent subsequence with limit in M. Hence if M is compact, then M is approximatively compact; but the converse is not true.

Theorem 1.7.10. [ANT77] *Let X be a uniformly convex metric linear space. Then any complete convex subset M of X is approximatively compact.*

Proof. Let $x \in X$ and $\{x_n\} \subseteq M$ such that $\lim_{n\to\infty} q(x - x_n) = \gamma = d(x, M)$, we show that $\{x_n\}$ is Cauchy sequence. Let $\varepsilon > 0$, take $r = \gamma > 0$. Take $\delta > 0$ as in the definition of uniformly convex of X. Since $q(x - x_n) \to \gamma = r$ in $\mathbb{R}$, taking $\varepsilon' = \delta$, then there exists an integer $N \geq 1$ such that for all $n \geq N$

$$|q(x - x_n) - \gamma| < \varepsilon' = \delta, \text{ or } -\delta < q(x - x_n) - \gamma < \delta, \text{ or } q(x - x_n) < \gamma + \delta. \quad (1)$$

Now, fix any $n, m \geq N$. Then, by(1),

$$q(x - x_n) < \gamma + \delta, \; q(x - x_m) < \gamma + \delta. \quad (2)$$

Since M is convex, $\frac{x_n + x_m}{2} \in M$ and so

$$\gamma = d(x, M) \leq q\left(x - \frac{x_n + x_m}{2}\right). \quad (3)$$

By uniformly convex of (X, q), (2) and (3), we have

$$\begin{aligned} q[(x - x_n) - (x - x_m)] &< \varepsilon, \\ \text{or } q(x_n - x_m) &< \varepsilon \qquad \text{for all } n, m \geq N, \end{aligned}$$

so $\{x_n\}$ is a Cauchy sequence in M. since M is complete $x_n \to y \in M$. Hence we have $\{x_n\}$ as a subsequence of $\{x_n\}$ which converges in M. Thus M is approximatively compact. □

Corollary 1.7.11. [ANT77] *Let (X, q) be a uniformly convex F-space. Then any closed convex subset M of X is approximatively compact.*

Proof. Here M is a closed subset of complete metric space X and so M is also complete. Hence, by above Theorem, M is approximatively compact. □

Corollary 1.7.12. [ANT77] Let (X, d) be a uniformly convex metric linear space. Then any complete convex subset M of X is proximinal (and hence Chebyshev).

Proof. Let $x \in X$, and $\gamma = d(x, M)$. By definition of $d(x, M)$, there exist a minimizing sequence $\{y_m\}$ in M such that

$$d(x, y_m) \to \gamma. \quad (1)$$

By Theorem 1.7.10, M is approximatively compact, so $\{y_m\}$ has subsequence $\{y_{m_k}\}$ such that

$$\lim_{k\to\infty} y_{m_k} = y \in M.$$

Now

$$d(x, y) = d(x, \lim_{k\to\infty} y_{m_k}) = \lim_{k\to\infty} d(x, y_{m_k}). \quad (2)$$

Then by (1) and (2) the sequence $\{d(x, y_m)\}$ and its subsequence $\{d(x, y_{m_k})\}$ both converge, so they must have the same limit $d(x, y) = \gamma = d(x, M)$. Then $y \in P_M(x)$, hence M is proximinal set. Since (X, d), being uniformly convex, is strictly convex, M is Chebyshev. $\square$

Corollary 1.7.13. [ANT77] *Let (X, q) be a uniformly convex F-space. Then any closed convex subset M of X is proximinal (and hence Chebyshev).*

Proof. Here M is a closed subset of complete metric space X and so M is also complete. Hence, by above corollary , M is proximinal (and hence Chebyshev). $\square$

Remark. It is well-known that every uniformly convex Banach space is a reflexive space. Therefore, the following results extends the proximinality of vector subspaces from uniformly convex to reflexive spaces.

Theorem 1.7.14. *Let $(X, ||\cdot||)$ be a reflexive Banach space. Then any closed vector subspace M of X is proximinal.*

Proof. Let $x \in X\backslash M$, and $\gamma = d(x, M)$. By definition of $d(x, M)$, there exist a minimizing sequence $\{y_m\}$ in M such that

$$\lim_{m\to\infty} ||x - y_m|| \to \gamma. \tag{1}$$

Since X is reflexive, its closed vector subspace M is also reflexive. Since $\{y_m\}$ is a bounded sequence in M, so it has subsequence $\{y_{m_k}\}$ such that

$$y_{m_k} \to y \;\text{ weakly in } M.$$

Then clearly $x - y_{m_k} \to x - y$ weakly in X; i.e. $f(x - y_{m_k}) \to f(x - y)$ in $\mathbb{R}$ for all $f \in X^*$. Now

$$\begin{aligned}
\gamma &= d(x, M) \le ||x - y|| = ||\widehat{x - y}|| = \sup\{|(\widehat{x - y})(f)| : f \in X^*, ||f|| \le 1\} \\
&= \sup\{|f(x - y)| : f \in X^*, ||f|| \le 1\} = \sup\{\lim_k |f(x - y_{m_k})| : f \in X^*, ||f|| \le 1\} \\
&\le \sup\{\liminf_k ||f|| \cdot ||x - y_{m_k}|| : f \in X^*, ||f|| \le 1\} \le \liminf_k ||x - y_{m_k}|| \\
&= \lim_k ||x - y_{m_k}|| = \gamma;
\end{aligned}$$

i.e., $d(x, y) = \gamma = d(x, M)$, and so $y \in P_M(x)$. Hence M is proximinal set. $\square$

Theorem 1.7.15. [Nar79] *A convex boundedly weakly compact set M in a strictly convex metric linear space (X, d) is proximinal, and hence Chebyshev.*

Proof. Let M be a convex boundedly weakly compact subset of X and x an arbitrary point of X. Then choose a sequence $\{y_n\}$ in M such that

$$\lim_{n\to\infty} d(y_n, x) = d(x, M). \tag{1}$$

Now, for any $n \ge 1$,

$$d(y_n, 0) \le d(y_n, x) + d(x, 0). \tag{2}$$

Since the sequence $\{d(y_n, x)\}$, being convergent in $\mathbb{R}$, is bounded, it follows from (2) that the sequence $\{y_n\}$ is bounded. By assumption, $\{y_n\}$ has a subsequence $\{y_{n_k}\}$

such that y_{n_k} is weakly convergent to z (say) in M. We claim that $d(x,z) = d(x,M)$. It is sufficient to prove that, for any $\varepsilon > 0$, $d(x,z) \leq d(x,M) + \varepsilon$. Since X is strictly convex, the closed ball $B = B[x, d(x,M) + \varepsilon]$ is convex (see last section). By the Mazur Theorem, the ball B, being convex and closed, is weakly closed. By (1), there exists $N \geq 1$ such that, for all $k \geq N$,

$$|d(y_{n_k}, x) - d(x,M)| \leq \varepsilon \text{ or that } d(y_{n_k}, x) \leq d(x,M) + \varepsilon.$$

Therefore, $\{y_{n_k} : k \geq N\} \subseteq B$. Since $y_{n_k} \to z$ weakly and B is weakly closed, we have $z \in B$, i.e, $d(z,x) \leq d(x,M) + \varepsilon$.] Thus $d(x,z) = d(x,M)$, and so $z \in P_M(x)$. □

Properties of the Proximity Map

Recall that: if X is a Hibert space and C a nonempty closed convex subset of X, then C is a Chebyshev set in X. Conseqyently, the proximity mapping $P_C : X \to C$ is a well-defined single-valued map on X.

Theorem 1.7.16. ([Deu01], p. 43, 72-73; [KK01], p. 135-136) *Let X be a Hibert space and C a nonempty closed convex subset of X. Then the proximity mapping $P_C : X \to C$ has the following properties:*

(1) *$P_C^2 = P_C$, i.e. $P_C(P_C(x)) = P_C(x)$ for all $x \in X$.*

(2) *For any $x \in X$,all $z \in C$,*

$$z = P_C(x) \textit{ iff } < x - z, z - y > \geq 0 \textit{ for all } y \in C.$$

(3) *$||P_C(x) - P_C(y)||^2 \leq < x - y, P_C(x) - P_C(y) >$ for all $x, y \in X$.*

(4) *$< x - y, P_C(x) - P_C(y) > \geq 0$ for all $x, y \in X$.*

(5) *$||P_C(x) - P_C(y)|| \leq ||x - y||$ for all $x, y \in X$ (nonexpansive).*

Proof. (1) Note that $P_C(y) = y$ for all $y \in C$. Now, for any $x \in X$, $P_C(x) \in C$, and so $P_C(P_C(x)) = P_C(x)$.

(2) ($\Rightarrow$) Suppose $z = P_C(x)$, but there exists some $y \in C$ such that

$$< x - z, z - y > \quad < 0, \textit{ or } < x - z, y - z > \quad > 0. \tag{1}$$

Since C is convex, for each $1 < t < 1$, $y_t = ty + (1-t)z) = z + t(y - z) \in [z, y] \subseteq C$. Since $z = P_C(x)$,

$$||x - z|| \leq ||x - y_t||. \tag{2}$$

Now,

$$\begin{aligned}
||x-y_t||^2 &= <x-y_t, x-y_t> \\
&= <x-ty-(1-t)z, x-ty-(1-t)z> \\
&= <x-z-t(y-z), x-z-t(y-z)> \\
&= <x-z, x-z> + <x-z, -t(y-z)> \\
+ &< -t(y-z), x-z> + < -t(y-z), -t(y-z)> \\
&= ||x-z||^2 - 2t <x-z, y-z> + t^2||y-z||^2 \\
&= ||x-z||^2 - t[2 <x-z, y-z> - t||y-z||^2 \\
&< ||x-z||^2 \text{ (using (1) and for } t \text{ sufficiently small)},
\end{aligned}$$

and so $||x-y_t|| < ||x-z||$, which contradicts (2).

On the other hand, Suppose $<x-z, z-y> \geq 0$ for all $y \in C$. Then for any $y \in C$,

$$\begin{aligned}
||x-z||^2 &= <x-z, x-z> = <x-z, x-y+y-z> \\
&= <x-z, x-y> + <x-z, y-z> \\
&= <x-z, x-y> - <x-z, z-y> \\
&\leq <x-z, x-y> \text{ (since } <x-z, z-y> \geq 0) \\
&\leq ||x-z||.||x-y||;
\end{aligned}$$

that is, $||x-z|| \leq ||x-y||$ for all $y \in C$, and so $z = P_C(x)$.

(3) For all $x, y \in X$,

$$\begin{aligned}
&< x-y, P_C(x)-P_C(y)> \\
&= <(x-P_C(x)) + (P_C(x)-P_C(y)) + (P_C(y)-y), P_C(x)-P_C(y)> \\
&= <x-P_C(x), P_C(x)-P_C(y)> + <P_C(x)-P_C(y), P_C(x)-P_C(y)> \\
+ &< P_C(y)-y, P_C(x)-P_C(y)> \text{ (using linearity in the first component)} \\
&= <x-P_C(x), P_C(x)-P_C(y)> + ||P_C(x)-P_C(y)||^2 + \\
&< y-P_C(y), P_C(y)-P_C(x)> \\
&\geq 0 + ||P_C(x)-P_C(y)||^2 + 0 \text{ (by (2) taking } z = P_C(x) \text{ and } z = P_C(y)). \\
&= ||P_C(x)-P_C(y)||^2
\end{aligned}$$

(4) For all $x, y \in X$, by (3),

$$<x-y, P_C(x)-P_C(y)> \geq ||P_C(x)-P_C(y)||^2 \geq 0.$$

(5) For all $x, y \in X$, by (3),

$$\begin{aligned}
||P_C(x) - P_C(y)||^2 &\leq\ < x - y, P_C(x) - P_C(y) > \\
&\leq\ ||x - y||.||P_C(x) - P_C(y)|| \quad \text{(by Cauchy-Shwartz inequality)} \\
\text{hence } ||P_C(x) - P_C(y)|| &\leq\ ||x - y||. \quad \square
\end{aligned}$$

Chapter 2

Classical Fixed Point Theorems in Metric and Banach Spaces

In this chapter we consider several famous and useful fixed point theorems. These include fixed point theorem for contraction mappings, fixed point theorems of Banach, Brouwer, Schauder, Tychonoff-Schauder, Markov-Kakutani and Kannan and fixed point theorem for contraction multivalued mappings.

2.1 Banach Contraction Principle and Related Fixed Theorems

In this section we present some important results of Banach (1922), Edelstein (1962), Caristi (1976) and Kirk (1965) in fixed point theory.

Banach's Contraction Mappings Principle is remarkable in its simplicity, yet it is perhaps the most widely applied fixed point theorem in all of analysis. This is because the contractive condition on the mapping is easy to test and it requires only a complete metric space for its setting. Although the basic idea was known to others earlier, the principle first appeared in explicit form in Banach's 1922 thesis [Ban22], where it was used to establish the existence of a solution to an integral equation.

Definition: Let X be a set and $K \subseteq X$, and let $T : K \to K$ be a function, called a **self-mapping** on K. An element $x_0 \in K$ is called a **fixed point** of T if $T(x_0) = x_0$, or, equivalently, x_0 is a solution of the equation $g(x) = T(x) - x = 0$.

We denote by $Fix(T)$ the set of all fixed points of T in K. In general, $Fix(T)$ may be an empty set.

Example. Let X be the two-element set $\{a, b\}$. The function $T : X \to X$, defined by $T(a) = b$ and $T(b) = a$ has no fixed point.

Remarks. (1) Note that the definition of a fixed point requires no structure on either the set K or the function T.

(2) Geometric interpretation of $Fix(T)$ is:

$$\begin{aligned} Fix(T) &= \{x \in X : T(x) = x\} \\ &= \{x \in X : y = T(x) \text{ and } y = x\} \\ &= \{x \in X : y = T(x)\} \cap \{x \in X : y = x\} \\ &= \text{ common solution of the equations } y = T(x) \text{ and } y = x. \end{aligned}$$

In particular, if $X = \mathbb{R}$ and $f : \mathbb{R} \to \mathbb{R}$ is function (e.g. $f(x) = \sin x$ or $\cos x$ or x^2), then $Fix(f)$ is the set of all $x \in \mathbb{R}$ such the graphs of $y = f(x)$ and $y = x$ intersect.

Problem: The main problem in Fixed Point Theory is to determine: Under what conditins on X, K and T,

(a) $Fix(T) \neq \varnothing$, or $Fix(T)$ is a sigleton (i.e., T has a *unique* fixed point in K).

(b) If K is a convex subset of a vector space (or a topological vector space) X, is $Fix(T)$ also convex ?

Now we define various types of "contractive" mappings used in Fixed Point Theorey.

Definition: Let (X, d) be a metric space. A mapping (or a function) $T : X \to X$ is called:

(i) a **contraction** if there exists a real number $0 \leq \alpha < 1$ such that

$$d(T(x), T(y)) \leq \alpha.d(x, y) \text{ for all } x, y \in X.$$

(ii) **contractive** if

$$d(T(x), T(y)) < d(x, y) \text{ for all } x, y \in X.$$

(iii) **nonexpansive** if

$$d(T(x), T(y)) \leq d(x, y) \text{ for all } x, y \in X.$$

(iv) K-**Lipschitz** if there exists a real number $K \geq 0$ such that

$$d(T(x), T(y)) \leq K.d(x, y) \text{ for all } x, y \in X.$$

Note. (1) T is a contraction $\Rightarrow$ T is contractive $\Rightarrow$ T is nonexpansive $\Leftrightarrow$ T is 1-Lipschitz:

$$d(T(x), T(y)) \leq \alpha.d(x, y) < d(x, y) \leq d(x, y) \text{ for all } x, y \in X.$$

(2) T is K-Lipschitz with $0 \leq K < 1$ means T is a contraction.

(3) T is K-Lipschitz $\Rightarrow$ T is uniformly continuous $\Rightarrow$ T is continuous.

Proof of (3). Suppose T is K-Lipschitz with $K \geq 0$, and let $\varepsilon > 0$. First suppose $K > 0$. Then taking $\delta = \frac{\varepsilon}{K}$, we have

$$d(T(x), T(y)) \leq K.d(x, y) < K.\frac{\varepsilon}{K} = \varepsilon \text{ for all } x, y \in X \text{ with } d(x, y) < \delta.$$

If $K = 0$, then taking any $\delta > 0$, we have

$$d(T(x), T(y)) \leq K.d(x, y) = 0 < \varepsilon \text{ for all } x, y \in X \text{ with } d(x, y) < \delta.$$

Hence in each case, T is uniformly continuous. (Note that if $K = 0$, then clearly $T(x) = T(y))$ for all $x, y \in X$, and so T is a constant function, which is trivially uniformly continuous.) □

(4) By (1) and (3), clearly every non-expansive mapping, hence every contractive or contraction mapping, is continuous.

Example. Let $f : \mathbb{R} \to \mathbb{R}$ be a differentiable real function. If there is a real number $\alpha < 1$ for which the derivative $f^{'}(x)$ satisfies

$$\left| f^{'}(x) \right| \leq \alpha \text{ for all } x \in \mathbb{R}.$$

Then f is a contraction with respect to the usual metric on $\mathbb{R}$.

Solution. Let $x, y \in \mathbb{R}$ and without loss of generality assume $x < y$. By the Mean Value Theorem, there exists $\xi \in (x, y)$ such that $f(y) - f(x) = f^{'}(\xi)(y - x)$ and therefore

$$|f(y) - f(x)| = \left| f^{'}(\xi) \right| |(y - x)| \leq \alpha \left| (y - x) \right|.$$

Hence f is a contraction on (a, b).

Definition: Let (X, d) be a complete metric space. For any fixed $x \in X$, the sequence $\{T^n(x)\}_{n=0}^{\infty}$ defined by

$$\{T^n(x)\}_{n=0}^{\infty} = \{x, T(x), T^{\,2}(x), T^{\,3}(x), ...\}.$$

is called the **Picard iteration** of x under T.

Theorem 2.1.1. (Banach Contraction Principle) [Ban22] *Let (X, d) be a complete metric space and let $T : X \to X$ be a contraction mapping. Then T has a unique fixed point x_0. In fact, each $x \in X$, the Picard sequence $\{T^n(x)\}_{n=0}^{\infty}$ converges to a point x_0 (say) in X which turns out to be the unique fixed point of T.*

Proof. (I) Since T is a contraction, $\exists$ an $\alpha \in [0, 1)$ such that

$$d(T(x), T(y)) \leq \alpha d(x, y) \;\forall x, y \in X. \tag{1}$$

Take any $x \in X$ and consider $\{T^n(x)\}_{n=0}^{\infty}$.

(II) $\{T^n(x)\}$ is a Cauchy sequence in X: First we prove that for $m < n$ where m, n are positive integers ,

$$d(T^{\,m}(x), T^{\,n}(x)) \leq \frac{\alpha^m}{1-\alpha} d(x, T\ (x))\ \forall x \in X.$$

By (1),

$$d(T(x), T^{\,2}(x)) \leq \alpha d(x, T(x))$$

For any integer $k \geq 1$,

$$\begin{aligned}
d(T^k(x), T^{k+1}(x)) &\leq \alpha d(T^{k-1}(x), T^k(x)) \\
&\leq \alpha.\alpha d(T^{k-2}(x), T^{k-1}(x)) \\
&\leq \text{..........} \\
&\leq \alpha.\alpha...\alpha(k\text{-time}) d(T^{k-k}(x), T^{k-(k-1)}(x)) \\
&= \alpha^k d(x, T(x)) \qquad (2)
\end{aligned}$$

Now, if $m < n$,

$$\begin{aligned}
d(T^m(x), T^n(x)) &\leq d(T^m(x), T^{m+1}(x)) + d(T^{m+1}(x), T^{m+2}(x)) \\
&\quad +........ + d(T^{n-1}(x), T^n(x)) \\
&\leq \alpha^m d(x, T(x)) + \alpha^{m+1} d(x, T(x)) + ... + \alpha^{n-1} d(x, T(x)) \\
&= (\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1}) d(x, T(x)) \quad [\text{by } (2)] \\
&\leq (\alpha^m + \alpha^{m+1} + ... + \alpha^{n-1} + \alpha^n + \alpha^{n+1} + ...) d(x, T(x)) \\
&= \frac{\alpha^m}{1-\alpha} d(x, T(x)).
\end{aligned}$$

Since $0 \leq \alpha < 1$, $\lim\limits_{m\to\infty} \alpha^m = 0$. Hence

$$\lim_{m\to\infty} \frac{\alpha^m}{1-\alpha} d(x, T(x)) = 0,$$

i.e., given $\varepsilon > 0, \exists$ an $N \geq 1$ such that $\left|\frac{\alpha^m}{1-\alpha} d(x, T(x)) - 0\right| = \frac{\alpha^m}{1-\alpha} d(x, T(x)) < \varepsilon\ \forall$ $m \geq N$. So

$$d(T^m(x), T^n(x)) < \varepsilon \quad \forall m, n \geq N.$$

Thus $\{T^n(x)\}$ is Cauchy in X.

(III) Since X is complete,

$$\lim_{n\to\infty} T^n(x) = x_0 \text{ (say) in } X.$$

Now, since T is continuous,

$$T(x_0) = T(\lim_{n\to\infty} T^n(x)) = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T^n(x) = x_0.$$

Thus x_0 is a fixed point of T.

(IV) **Uniqueness of** x_0: From (III), $T(x_0) = x_0$. Suppose also that $\exists\ y_0 \in X$ such that $T(y_0) = y_0$. Then

$$\begin{aligned} d(x_0, y_0) &= d(T(x_0), T(y_0)) \\ &< \alpha d(x_0, y_0) \\ &< d(x_0, y_0) \text{ (since } \alpha < 1), \end{aligned}$$

a contradiction. Hence x_0 is a unique fixed point of T. □

Examples

Example (1) Let $X = [0, \frac{1}{4}]$, a complete metric space and $T : [0, \frac{1}{4}] \to [0, \frac{1}{4}]$ such that

$$T(x) = x^2, \;\; x \in [0, \frac{1}{4}].$$

Then: (i) T is a contraction;

(ii) T has a unique fixed point.

Solution. (i) For any $x, y \in [0, \frac{1}{4}]$,

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| = \left|x^2 - y^2\right| = |x + y|\,.|x - y| \\ &\leq (|x| + |y|).|x - y| \leq (\frac{1}{4} + \frac{1}{4}).|x - y| \\ &\leq \frac{1}{2}.|x - y| = \alpha.|x - y| \quad (\alpha = \frac{1}{2}). \end{aligned}$$

Hence T is a contraction with $0 \leq \alpha < 1$.

(ii) Clearly, T has a unique fixed point $x_0 = 0$. □

Example (2) Let $X = (0, \frac{1}{4}]$, an incomplete metric space, and $T : (0, \frac{1}{4}] \to [0, \frac{1}{4}]$ such that

$$T(x) = x^2, \;\; x \in (0, \frac{1}{4}].$$

Then: (i) T is a contraction;

(ii) T has no fixed point.

Solution. (i) For $x, y \in (0, \frac{1}{4}]$,

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| = \left|x^2 - y^2\right| = |x + y|\,.|x - y| \\ &\leq (|x| + |y|).|x - y| \leq (\frac{1}{4} + \frac{1}{4}).|x - y| \\ &\leq \frac{1}{2}.|x - y| = \alpha.|x - y| \quad (\alpha = \frac{1}{2}). \end{aligned}$$

Hence T is a contraction with $\alpha = \frac{1}{2}$, $0 \leq \alpha < 1$.

(ii) Clearly, T has no fixed in $X = (0, \frac{1}{4}]$. □

Example (3) Let $X = [0, 1]$, a complete metric space, and $T : [0, 1] \to [0, 1]$ such that

$$T(x) = x^2, \;\; x \in [0, 1].$$

Then: (i) T is K-Lipshitz, not a contraction or nonexpansive;

(ii) T has two fixed points (not a unique one).

Solution. (i) For any $x, y \in [0, 1]$,

$$\begin{aligned} d(Tx, Ty) &= |Tx - Ty| = \left|x^2 - y^2\right| = |x + y| \,.|x - y| \\ &\leq (|x| + |y|).|x - y| \leq (1 + 1).|x - y| \\ &\leq 2.|x - y|. \end{aligned}$$

Therefore, T is K-Lipshitz (with $K = 2$), but not a contraction or nonexpansive.

(ii) Clearly, T has two fixed points $x_0 = 0$ and $y_0 = 1$. □

Example (4) Let $X = \mathbb{R}$, a complete metric space, and $T : \mathbb{R} \to \mathbb{R}$ such that

$$T(x) = x + 1, \;\; x \in \mathbb{R}.$$

Then: (i) T is non-expansive (or 1-Lipshitz), not a contraction;

(ii) T has no fixed points.

Solution. (i) For any $x, y \in \mathbb{R}$,

$$d(Tx, Ty) = |Tx - Ty| = |x + 1 - y - 1| = |x - y|.$$

Therefore, T is non-expansive (or 1-Lipshitz), but not a contraction.

(ii) Clearly, T has no fixed points. □

Remark. It seems intuitively clear that any continuous function mapping the unit interval into itself will have a fixed point, but the Banach Theorem applies only to functions f that satisfy $\left|f'(x)\right| \leq \alpha$ for some $\alpha < 1$. An elementary example of this is the function $f(x) = 1 - x$, which has an obvious fixed point at $x = 1/2$, but whose derivative satisfies $\left|f'(x)\right| = 1$ everywhere, therefore

$$d(f(x), f(x')) = d(x, x'),$$

so f is not a contraction and the Banach Fixed point Theorem does not apply to f. The fixed point theorem due to Brouwer (Section 2.2) covers this case as well as a great many others that the Banach Theorem fails to cover because the relevant functions are not contractions.

Definition: Let $K \subseteq (X, d)$. A mapping $T : K \to K$ is said to be a **Banach operator** if there exists a constant α such that $0 \leq \alpha < 1$ and for each $x \in K$

$$d(T^2x, Tx) \leq \alpha d(Tx, x). \tag{BO}$$

Every contraction mapping is a Banach operator, but not conversely. A Banach operator need not be continuous, nor need its fixed points be unique.

Theorem 2.1.2. [Sub77, KhKh95] *Let C be a closed subset of a metric space (X, d), and let $T : C \to C$ be a continuous Banach operator. Then T has a fixed point in C each of the following cases:*

(a) (X, d) *is complete.*

(b) $\overline{T(C)}$ *is compact.*

Proof. (Outline) These are similar to that of the Banach Contraction Principle, with slight modification, as follows.

(a) (X, d) is complete. Take any $x \in C$ and consider $\{T^n(x)\}_{n=0}^{\infty}$. Then, using (BO), for any integer $k \geq 1$,

$$\begin{aligned} d(T^k(x), T^{k+1}(x)) &\leq \alpha d(T^{k-1}(x), T^k(x)) \\ &\leq \ldots\ldots\ldots \\ &\leq \alpha^k d(x, T(x)). \end{aligned}$$

Now, if $m < n$,

$$\begin{aligned} d(T^m(x), T^n(x)) &\leq d(T^m(x), T^{m+1}(x)) + d(T^{m+1}(x), T^{m+2}(x)) \\ &\quad + \ldots\ldots + d(T^{n-1}(x), T^n(x)) \\ &\leq \frac{\alpha^m}{1-\alpha} d(x, T(x)). \end{aligned}$$

Since $0 \leq \alpha < 1$, $\lim_{m\to\infty} \alpha^m = 0$. Hence $\{T^n(x)\}$ is Cauchy in $T(C)$. Since C is closed in X and X is complete, C is also complete and so

$$\lim_{n\to\infty} T^n(x) = x_0 \text{ (say) exists and } x_0 \in \overline{T(C)} \subseteq \overline{C} = C \text{ (since } C \text{ is closed).}$$

Now, since T is assumed to be continuous,

$$T(x_0) = T(\lim_{n\to\infty} T^n(x)) = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T^n(x) = x_0.$$

Thus x_0 is a fixed point of T in C.

(b) Suppose $\overline{T(C)}$ is compact. Take any $x \in C$ and consider $\{T^n(x)\}_{n=0}^{\infty}$. As in the above proof, if $m < n$,

$$d(T^m(x), T^n(x)) \leq \frac{\alpha^m}{1-\alpha} d(x, T(x)).$$

Since $0 \leq \alpha < 1$, $\lim_{m\to\infty} \alpha^m = 0$. Hence $\{T^n(x)\}$ is Cauchy in $T(C)$. Since closure $\overline{T(C)}$ is compact in the metric space, it is complete and so

$$\lim_{n\to\infty} T^n(x) = x_0 \text{ (say) exists and } x_0 \in \overline{T(C)} \subseteq \overline{C} = C \text{ (since } C \text{ is closed).}$$

Then, by continuity of T,

$$T(x_0) = T(\lim_{n\to\infty} T^n(x)) = \lim_{n\to\infty} T^{n+1}(x) = \lim_{n\to\infty} T^n(x) = x_0.$$

Thus x_0 is a fixed point of T in C. $\square$

Next we consider more general notions of the contractive mappings and Caristi mappings and present famous fixed point theorems of Edelstein (1962) and Caristi (1976).

Recall that: a mapping $T : (X, d) \to (X, d)$ is called **contractive** if

$$d(Tx, Ty) < d(x, y) \text{ for all } x, y \in X.$$

Theorem 2.1.3. (Edelstein, 1962) *Let (X, d) be a compact metric space and $T : X \to X$ is a contractive mapping. Then T has a unique fixed point in X.*

Proof. Define $\varphi : X \to \mathbb{R}^+$ by

$$\varphi(x) = d(x, Tx), \ \ x \in X.$$

Then φ is continuous (as d, T are continuous). [In fact, if $x_n \to x$, Then

$$\lim_{n\to\infty} \varphi(x_n) = \lim_{n\to\infty} d(x_n, Tx_n) = d(x, Tx) = \varphi(x).]$$

Since X is compact, φ attains its infimum on X at some point x_0 (say). i.e.

$$\varphi(x_0) = \inf_{x\in X} \varphi(x).$$

We show that $T(x_0) = x_0$. Suppose $T(x_0) \neq x_0$. Then

$$\varphi(T(x_0)) = d(T(x_0), T^2(x_0)) < d(x_0, T(x_0)) = \varphi(x_0),$$

so

$$\varphi(T(x_0)) < \varphi(x_0) = \inf_{x\in X} \varphi(x).$$

This is a contradiction (since $T(x_0) \in X$). Hence $T(x_0) = x_0$. To show the uniqueness of x_0, suppose $\exists$ a $y_0 \in X$ such that $T(y_0) = y_0$. Then

$$\begin{aligned} d(x_0, y_0) &= d(Tx_0, Ty_0) \\ &< d(x_0, y_0) \text{ (since } T \text{ is contractive)}, \end{aligned}$$

a contradiction. So $x_0 = y_0$. $\square$

Definition: Let (X, d) be a metric space. A map $T : X \to X$ is called a **Caristi map** (or **Caristi-Ekeland map**) if $\exists$ a lower semi continuous (l.s.c.) $\varphi : X \to \mathbb{R}^+$ such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \ \forall x \in X.$$

Lemma 2.1.4. *Any contraction map $T : (X, d) \to (X, d)$ is a Caristi map.*

Proof. Since T is a contraction map, $\exists$ an $\alpha \in [0, 1)$ such that

$$d(Tx, Ty) \leq \alpha d(x, y) \forall x, y \in X.$$

Define $\varphi : X \to \mathbb{R}^+$ by $\varphi(x) = \frac{1}{1-\alpha} d(x, Tx), x \in X$. Then φ is l.c.s. We need to show that $d(x, Tx) \leq \varphi(x) - \varphi(Tx)$ $\forall$ $x \in X$. First

$$\begin{aligned} \varphi(Tx) &= \frac{1}{1-\alpha} d(x, T^2x) \leq \frac{1}{1-\alpha}.\alpha.d(x, Tx) \\ &= \alpha.\frac{1}{1-\alpha} d(x, Tx) = \alpha.\varphi(x). \end{aligned}$$

Thus $-\alpha.\varphi(x) \leq -\varphi(Tx)$.Now, using (1),

$$d(x, Tx) = (1-\alpha)\varphi(x) = \varphi(x) - \alpha.\varphi(x) \leq \varphi(x) - \varphi(Tx). \quad \square$$

Theorem 2.1.5. **(**Caristi, 1976) *Let (X, d) be a complete metric space and $T : X \to X$ is a Caristi map. Then T has a fixed point in X.*

Proof. (I) Since T is a Caristi map, $\exists$ a l.s.c. $\varphi : X \to \mathbb{R}^+$ such that

$$d(x, Tx) \leq \varphi(x) - \varphi(Tx) \forall x \in X. \tag{1}$$

Define a partial order $\leq$ on X by:

$$x \leq y \Longleftrightarrow d(x, y) \leq \varphi(x) - \varphi(y).$$

By definition of $\leq$, putting $y = Tx$ then

$$x \leq Tx \Longleftrightarrow d(x, Tx) \leq \varphi(x) - \varphi(Tx), \text{ which is true by (1).}$$

Thus, $x \leq Tx$. Next in steps (II)-(III) using Zorn's lemma we show that $\exists$ an $x_0 \in X$ such that $Tx_0 = x_0$.

(II) Let M be a maximal totally ordered subset of X. Take any totally ordered subset $\{x_\alpha : \alpha \in I\}$ of M. Then we need to show that $\{x_\alpha\}_{\alpha \in I}$ is a Cauchy net in X. First note that $\{\varphi(x_\alpha) : \alpha \in I\}$ is a decreasing net in $\mathbb{R}^+$. Indeed

$$\begin{aligned} \alpha &\leq \beta \Longleftrightarrow x_\alpha \leq x_\beta \\ &\Longleftrightarrow 0 \leq d(x_\alpha, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta) \\ &\Longleftrightarrow \varphi(x_\beta) \leq \varphi(x_\alpha) \\ &\Longleftrightarrow \varphi(x_\alpha) \geq \varphi(x_\beta). \end{aligned}$$

Thus

$$r = \inf_{\alpha \in I}\{\varphi(x_\alpha)\} \text{ exists in } \mathbb{R}^+,$$

i.e. given $\varepsilon > 0, \exists$ an $\alpha_0 \in I$ such that

$$r \leq \varphi(x_\alpha) < r + \varepsilon \ \forall \alpha \geq \alpha_0.$$

Hence, if $\beta > \alpha \geq \alpha_0$,

$$d(x_\alpha, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta) \leq \varphi(x_\alpha) - r < (r + \varepsilon) - r = \varepsilon.$$

Then $\{x_\alpha\}_{\alpha \in I}$ is Cauchy in X.

(III) Since X is complete, $x_\alpha \to x_0$ (say) in X. Since $\varphi : X \to \mathbb{R}^+$ is l.s.c.

$$\varphi(x_0) \leq \liminf_{\alpha} \varphi(x_\alpha) = r. \tag{2}$$

Now if $\beta > \alpha$ (or $\alpha < \beta$ or $x_\alpha < x_\beta$),

$$d(x_\alpha, x_\beta) \leq \varphi(x_\alpha) - \varphi(x_\beta).$$

Taking $\lim_{\beta} \inf$ and using (2),

$$d(x_\alpha, x_0) \leq \varphi(x_\alpha) - r \leq \varphi(x_\alpha) - \varphi(x_0).$$

Thus

$$x_\alpha \leq x_0 \text{ (by definition of partial order } \leq \text{).} \tag{3}$$

Since M is maximal and $\{x_\alpha\}_{\alpha \in I} \subseteq M$, so $x_\alpha \to x_0 \in M$. Since $d(x, Tx) \leq \varphi(x) - \varphi(Tx) \ \forall \ x \in X$. Then

$$x \leq Tx \ \forall x \in X \Longrightarrow x_0 \leq Tx_0. \tag{4}$$

By (3) & (4), $x_\alpha \leq x_0 \leq Tx_0$. But x_0 is an upper bound of $\{x_\alpha\}_{\alpha \in I}$, we have $Tx_0 \leq x_0$. Hence$Tx_0 = x_0$, so x_0 is a fixed point of T. □

Next, we define some classes of mappings related to non-expansive mappings:

Definition: Let (X, d) is a metric space and $K \subseteq X$. Then a mapping $T : K \longrightarrow K$ is called:

(i) **nonexpansive** if

$$d(T(x), T(y)) \leq d(x, y) \text{ holds for all } x, y \in K.$$

(ii) **asymptotically nonexpansive** (Goebel-Kirk [GK72]) if, for each pair $x, y \in K$:

$$d(T^n(x), T^n(y)) \leq k_n d(x, y),$$

where $(k_n)_{n \in N}$ is a sequence of real such that $\lim_{n \to \infty} k_n = 1$.

(iii) **quasi-nonexpansive** (Dotson [Do72]) if $Fix(T)$ is non-empty and

$$d(T(x), p) \leq d(x, p) \text{ for all } x \in K, \, p \in Fix(T).$$

Note. T is nonexpansive with $Fix(T) \neq \varnothing \Rightarrow T$ is quasi-nonexpansive:

$$d(T(x), p) \leq d(T(x), T(p)) \leq d(x, p) \text{ for all } x \in K,\ p \in Fix(T).$$

Properties of Fix(T)

Lemma 2.1.6. *Let $K \subseteq (X, d)$ and $T : K \to K$ is either (i) nonexpansive map, or (ii) quasi-non-expansive map. Then $Fix(T)$ is closed.*

Proof. If $Fix(T)$ is empty or a finite set, it is clearly closed. Suppose $Fix(T)$ is an infinite set. To prove that $Fix(T)$ is closed, it suffices to show that $\overline{Fix(T)} \subseteq Fix(T)$.Let $x \in \overline{Fix(T)}$, then there is a sequence $\{x_n\} \subseteq Fix(T)$ such that $x_n \to x$.

(i) Suppose T is nonexpansive. Then T is continuous, and so

$$\begin{aligned} T(x) &= T(\lim_{n\to\infty} x_n) = \lim_{n\to\infty} T(x_n) \quad \text{(by continuity of } T\text{)} \\ &= \lim_{n\to\infty} x_n \quad (\text{since } \{x_n\} \subseteq Fix(T)) \\ &= x. \end{aligned}$$

So $x \in Fix(T)$. Hence $\overline{Fix(T)} \subseteq Fix(T)$ i.e., $Fix(T)$ is closed.

(ii) Suppose T is quasi-nonexpansive. Since $x_n \to x$, for each $\varepsilon > 0$, there is $N \in \mathbb{N}$ such that $d(x, x_N) < \frac{\varepsilon}{2}$. Since T is quasi-nonexpansive and $x_N \in Fix(T)$,

$$\begin{aligned} d(x, T(x)) &\leq d(x, x_N) + d(T(x), x_N) \\ &\leq d(x, x_N) + d(x, x_N) \\ &\leq 2d(x, x_N) < 2\frac{\varepsilon}{2} = \varepsilon. \end{aligned}$$

Since ε is arbitrary, we must have $d(x, T(x)) = 0$; i.e., $x \in Fix(T)$. Therefore, $Fix(T)$ is closed. $\square$

Recall that: A normed space $(X, \|.\|)$ is called **strictly convex** if, for any $x, y \in X$ with $\|x\| = 1$, $\|y\| = 1$,

$$\|x + y\| = 2 \Rightarrow x = y.$$

Theorem 2.1.7. ([GK90], p. 34) *Let K be a convex closed subset of a strictly convex Banach space $(X, ||.||)$ and $T : K \longrightarrow K$ nonexpansive. Then the set $Fix(T)$ is a closed and convex.*

Proof. As proved above, if $K \subseteq (X, d)$, any metric space, and $T : K \longrightarrow K$ is nonexpansive or quasi-nonexpansive, then $Fix(T)$ is closed.

To prove that $Fix(T)$ is a convex set, choose any two points $x \neq y \in Fix(T)$ and $t \in (0, 1)$, and let

$$z = (1 - t)x + ty.$$

Since $T(x) = x, T(y) = y$ and T is nonexpansive,

$$\begin{aligned} ||x - T(z)|| + ||T(z) - y|| &= ||T(x) - T(z)|| + ||T(z) - T(y)|| \\ &\leq ||z - x|| + ||z - y|| \\ &= ||x - tx + ty - x|| + ||x - tx + ty - y|| \\ &= t.||x - y|| + (1 - t).||x - y|| \\ &= ||x - y|| \\ &\leq ||x - T(z)|| + ||T(z) - y||; \end{aligned}$$

hence

$$\begin{aligned} ||x - T(z)|| + ||T(z) - y|| &= ||x - y||, \\ ||x - z|| + ||z - y|| &= ||x - y|| \end{aligned}$$

.It follows that $x, T(z), y$ are colinear, while

$$||x - z|| = ||x - T(z)|| \text{ and } ||y - z|| = ||y - T(z)||.$$

Since X is strictly convex, it follows that $T(z) = z$. Hence $z = (1-t)x + ty \in Fix(T)$, and so $Fix(T)$ is convex. □

Note. The above result is also true for broader classes of mappings: quasi-nonexpansive and asymptotically nonexpansive mappings.

The following theorem is a famous result in Fixed Point Theory obtained independently by Kirk, Gohde and Browder in 1965.

Theorem 2.1.8. *Let K be a closed bounded convex subset of a uniformly convex Banach space X. Then every nonexpansive mapping $T : K \longrightarrow K$ has at least one fixed point.*

In fact, Kirk [Kir65] proved a slightly more general result than the above. This requires the terminology of "normal structure".

Definition: (Brodskii-Milman [BM48]) (i) ([GK90], p. 38-39; [KK01], p. 200) A convex set K in a Banach space X is said to have **normal structur**e if for each bounded, closed and convex subset $H \subseteq K$ for which $\delta(H) = \sup\{d(x, y) : x, y \in H\} > 0$ (e.g. H contains more than one point), there is some point $x_0 \in H$ such that

$$\sup\{d(x_0, x) : x \in H\} < \delta(H).$$

Such a point $x_0 \in H$ is called a **nondiametrical point** of H.

(ii) ([KK01], p. 200) A convex set K in a Banach space X is said to have **uniform normal structur**e if there exists a constant $c < 1$ such that, for each bounded, closed and convex subset $H \subseteq K$ for which $\delta(H) > 0$, there is some point $x_0 \in H$ such that

$$\sup\{d(x_0, x) : x \in H\} < c.\delta(H).$$

Theorem 2.1.9. ([KK01], p. 202-204) (a) *Every compact subset K of a Banach space X has normal structure. In particular, every finite dimensional Banach space has normal structure.*

(b) *Every uniformly convex Banach space X has uniform normal structure.*

(c) *If a Banach space space X has uniform normal structure, then it is reflexive.*

Recall that a subset K of a normed space X is **compact** (resp. **weakly compact**) if every sequence in K has a norm convergent (resp. weakly convergent) subsequence with limit in K.

Theorem 2.1.10. ([GK90], p. 40; [KK01], p. 203) *Let K be weakly compact convex subset of a Banach space X, and suppose K has normal structure. Then every nonexpansive mapping $T : K \longrightarrow K$ has at least one fixed point.*

2.2 Brouwer and Tychonoff-Schauder's Fixed Point Theorems for Continuous Mappings

In this section we take up famous fixed point theorems of Brouwer (1910), Schauder (1930), Tychonoff-Schauder (1935), Markov-Kakutani (1941) and Tarski (1955).

Theorem 2.2.1. (Brouwer, 1910**)** Let K be a nonempty, bounded closed (equivalently, compact) convex subset of $\mathbb{R}^n$. Every continuous function $T : K \to K$ has a fixed point.

Remark. The Brouwer Theorem requires only that f be continuous, not that it be a contraction, so there are lots of situations in which the Brouwer Theorem applies but the Banach Theorem doesn't. In particular. Brouwer's Theorem confirms our intuition that any continuous function mapping $[0, 1]$ into itself has a fixed point, not just the functions that satisfy $\left|f'(x)\right| \leq \alpha$ for some $\alpha < 1$.But conversely, the Banach Theorem doesn't require compactness or convexity, in fact, it doesn't require that the domain of f be a subset of a vector space, as this version of Brouwer's Theorem does. So there are also lots of situations where Banach's Theorem applies and Brouwer's doesn't.

There are several proofs of Brouwer's Theorem in literature, all of them require some highly specialized mathematical ideas.

Remark. ([KK01], p. 179) Brouwer's Theorem cannot be extended to an infinite dimensional Banach space.

Example. ([KK01], p. 179) Let $X = c_0$ with the norm (i.e. ℓ_∞-norm since $c_0 \subseteq c \subseteq \ell_\infty$)

$$||x|| = \sup\{|x_1|, |x_2|, |x_3|, ...),\ \ x = \{x_n\} \in c_0..$$

and $B = B[0,1]]$ the closed unit ball in X. Define $T : B \to B$ by

$$T(x) = (1 - |x_1|, x_1, x_2, ...), \ \ x = \{x_n\} \in B.$$

Then

$$\begin{aligned} ||T(x) - T(y)|| &= ||(1 - |x_1| - 1 + |y_1|, x_1 - y_1, x_2 - y_2, ...)|| \\ &= \sup\{| \ |x_1| - |y_1| \ |, |x_1 - y_1|, \ |x_2 - y_2|, ...\} \\ &= \sup\{|x_1 - y_1|, \ |x_2 - y_2|, ...\}, \text{ since } | \ |x_1| - |y_1| \ | \leq |x_1 - y_1| \\ &= ||x - y||, \end{aligned}$$

so T is nonexpansive, hence continuous on B. However,

$$\begin{aligned} T(x) &= x \Rightarrow (1 - |x_1|, x_1, x_2, ...) = (x_1, x_2, ...) \\ &\Rightarrow (1 - |x_1| - x_1, x_1 - x_2, x_2 - x_3, ...) = (0, 0, ...) \\ &\Rightarrow 1 - |x_1| - x_1 = 0, x_1 - x_2 = 0, x_2 - x_3 = 0, ... \\ &\Rightarrow 1 - |x_1| - x_1 = 0, x_1 = x_2 = x_3 = x_4 = ... \end{aligned}$$

Hence

$$\{x_n\} = \{x_1, x_2, x_3, ...\} = \{x_1, x_1, x_1, ...\} \to \{x_1\} \text{ (clearly);}$$

but $\{x_n\} \to 0$ (as $\{x_n\} \in c_0$), and so $x_1 = 0$. Therefore

$$1 - |x_1| - x_1 = 0 \Rightarrow 1 = 0,$$

a contradiction. Hence T has no fixed point in B. □

Next is a generalization of Brouwer's Theorem to normed vector spaces (which need not be finite-dimensional, as Brouwer's Theorem requires).

Theorem 2.2.2. (Schauder) ([KK01], p. 179) *Let K be a nonempty, compact, convex subset of a normed vector space X. Then every continuous function $T : K \to K$ has a fixed point.*

This theorem would apply, for example, to any compact convex subset of the normed space $C\,[0,1]$ with the max norm.

Theorem 2.2.3. (Schauder-Tychonoff, 1935) *Let K be a nonempty, compact, convex subset of a locally convex space X. Then every continuous function $T : K \to K$ has a fixed point.*

Theorem 2.2.4 (Markov-Kakutani, 1941) *Let K be a nonempty, compact, convex subset of a TVS X. Then every continuous linear map $T : K \to K$ has a fixed point. More generally, if $\mathcal{F}$ is a commuting family of continuous linear maps from $K \to K$, then F has a common fixed point in K.*

The proofs of above results some deep results such as KKM Theorem and Ky Fan Theorem. However, if T is assumed to be nonexpansive, then a simple proof of the following theorem is given by Dotson and Mann (American Math. Monthly (1977)).

Theorem 2.2.5. [DM77] *Let C be a compact convex subset of a normed space X and $T : C \to C$ a nonexpansive mapping. Then T has a fixed point in C.*

Proof. Step (I) For any fixed $p \in C$ and any fixed positive integer n, define a function $G_n : C \to C$ by

$$G_n(x) = \frac{1}{n+1}.p + \frac{n}{n+1}.Tx \text{ for all } x \in C.$$

Observe that $G_n(C) \subseteq C$ (since $G_n(x)$ is a convex combination of $p, Tx \in C$ and C is convex) Next observe, for any $x, y \in C$,

$$\begin{aligned}\|G_n(x) - G_n(y)\| &= \frac{\|p + nTx - p - nTy\|}{n+1} = \frac{n}{n+1}\|T(x) - T(y)\| \\ &\leq \frac{n}{n+1}\|x - y\| = k_n. \|x - y\|, \text{ (T is nonexpansive)}\end{aligned}$$

where $k_n = \frac{n}{n+1}$, $0 < k_n < 1$. So each G_n is a contraction on the complete metric space C. Hence, by the Banach Contraction Principle, each G_n has a unique fixed point x_n (say) in C.

Step (II) Now $\{x_n\} \subseteq C$, a compact metric space, and so $\{x_n\}$ has a convergent subsequence, also denoted by $\{x_n\}$ such that $x_n \to x_0$ (say) in C (since C is also closed). Now, for each n,

$$x_n = G_n(x_n) = \frac{1}{n+1}.p + \frac{n}{n+1}.Tx_n;$$

hence, by continuity of T,

$$\begin{aligned}x_0 &= \lim_{n\to\infty} x_n = \lim_{n\to\infty}\frac{1}{n+1}.p + \lim_{n\to\infty}\frac{n}{n+1}. \lim_{n\to\infty} T(x_n) \\ &= \lim_{n\to\infty}\frac{1}{n+1}.p + \lim_{n\to\infty}\frac{n}{n+1}. \lim_{n\to\infty} T(\lim_{n\to\infty} x_n) \\ &= o.p + 1.T(x_0) = T(x_0)\end{aligned}$$

Thus x_0 is a fixed point of T. □

Tarski Fixed Point Theorem

Definition: A non-empty set X with a relation $\leq$ is said to be a **partially ordered set** whenever the following conditions are satisfied.

(i) $x \leq x$ for every $x \in X$,

(ii) $x \leq y$ and $y \leq x$ implies that $x = y$, and

(iii) $x \leq y$ and $y \leq z$ implies that $x \leq z$.

If, in addition, for any two elements $x, y \in X$ either $x \leq y$ or $y \leq x$,then X is called a **totally ordered set**.

Definition: Let A be a subset of an partially ordered set X.

(i) $x \in X$ is called an **upper bound** of A if $y \leq x$ for every $y \in A$.

(ii) $z \in X$ is called a **lower bound** of A if $y \geq z$ for all $y \in A$.

Definition: An partially ordered set $(X, \leq)$ is called a **lattice** if any two elements $x, y \in X$ have a least upper bound denoted by $x \vee y = \sup(x, y)$ and a greatest lower bound denoted by $x \wedge y = \inf(x, y)$.

Definition: A partially ordered set $(X, \leq)$ is a **complete lattice** if every $S \subseteq X$ has a least upper bound and greatest lower bound in X.

Note. (1) A complete lattice is automatically a lattice.

(2) A complete lattice must be bounded, since one can always take $S = X$.

(3) A complete lattice need not be complete in the metric space sense even when X is a metric space.

(4) If X is complete lattice, then for any $a, b \in X$, $a \leq b.$, the interval $[a, b] = \{x \in X : a \leq x \leq b\}$ is a complete lattice.

Example 1. The set $X = [0, \frac{1}{2}) \cup \{1\}$ is a complete lattice, In particular, the least upper bound of $S = [0, \frac{1}{2})$ is $1 \in X$ (because the only upper bound of S is 1). But X is not complete in the metric space sense.

Given a partially ordered set X, an **interval** in X is a set of the form $[a, b] = \{x \in X : a \leq x \leq b\}$, where $a, b \in X$, $a \leq b$.

Example 2. Suppose that $X = [0, \frac{1}{2}) \cup (\frac{3}{4}, 1]$. Then $I = X$ is an interval in X that is not a complete lattice, since $S = [0, \frac{1}{2})$ has no least upper bound in I.

Definition: A mapping $T : X \to X$ is called **order preserving (or weakly increasing)** if $x \leq y$ implies $T(x) \leq T(y)$.

Again, since X is only partially ordered, there may be many pairs of x and y for which this property need not hold.

Theorem 2.2.6.(Tarski [Tar55]). *Let X be a non-empty complete lattice. If $T : X \to X$ is order preserving, then the set $Fix(T)$ is a non-empty complete lattice.*

Remarks. (1) A kind of converse of this theorem was proved by Anne C. Davis: *If every order preserving function $T : X \to X$ on a lattice T has a fixed point, then T is a complete lattice.*

(2) Since complete lattices cannot be empty, the theorem in particular guarantees the existence of at least one fixed point of T, and even the existence of a least (or greatest) fixed point. In many practical cases, this is the most important implication of the theorem.

(3) The **least fixed point** of T is the least element x such that $T(x) = x$, or, equivalently, such that $T(x) \leq x$; the dual holds for the greatest fixed point, the greatest element x such that $T(x) = x$.

2.3 Fixed Point Theorems for Kannan type Mappings

The classical Banach's contraction principle (1922) is no doubt one of the most useful results in fixed point theory. This theorem has many applications, but suffers from one drawback - the definition requires that the map $T : X \to X$ be continuous throughout the metric space (X, d). In 1968 Kannan [Kan68] gave an example of a contractive definition that does not require the continuity of T. There then followed a flood of papers involving contractive conditions , many of them do not require the continuity of T (see, for example, Chatterjea (1972), Zamfirescu (1972), Ciric (1974)). They obtained fixed point theorems using convergence for Picard iterates to fixed Points. Also Rhoades (1974, 1976) and Berinde (2005, 2007) obtained convergence for Mann iterates of certain mappings to fixed Points in Banach spaces.

First we present a useful result of Rhoades [Rh88]. This demonstrates that many contractive definitions require T be continuous at a fixed point. For this, it will be sufficient to show that such is the case for the most general definitions.

Theorem 2.3.1. ([Rh88], p. 233-234) *Let (X, d) be a complete metric space, $0 \leq k < 1$. Suppose that for each $x, y \in X$,*

$$d(Tx, Ty) \leq k. \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \qquad (1)$$

Then T is continuous at the fixed point x_0.

Proof. By a result of Rhoades ([Rh77], Theorem 11) the sequence

$$\{x_n\}_{n=0}^{\infty} = \{T^n(x_0)\}_{n=0}^{\infty} = \{x_0, T(x_0), T^{\,2}(x_0), T^{\,3}(x_0), ...\}.$$

of Picard iteration converges to the unique fixed point x_0. Then

$$\lim_{n\to\infty} Tx_n = \lim_{n\to\infty} T(T^n(x_0)) = \lim_{n\to\infty} T^{n+1}(x_0) = \lim_{n\to\infty} x_{n+1} = x_0,$$

and so $Tx_n \to x_0$.

To show that T is continuous at x_0, let $\{y_n\}$ be any sequence in X converging to x_0.Using (1),

$$\begin{aligned} d(Ty_n, Tx_n) &\leq k. \max\{d(x_n, y_n), d(y_n, Ty_n), d(x_n, Tx_n), d(y_n, Tx_n), d(x_n, Ty_n)\} \\ &\leq k. \max\{d(x_n, y_n), d(y_n, Tx_n) + d(Tx_n, Ty_n), \\ &\quad d(x_n, Tx_n), d(y_n, Tx_n), d(x_n, Tx_n) + d(Tx_n, Ty_n)\} \\ &\leq k. \max\{d(x_n, y_n), d(y_n, Tx_n) + d(Tx_n, Ty_n), d(x_n, Tx_n) + d(Tx_n, Ty_n)\}; \end{aligned}$$

or,

$$(1 - k)d(Ty_n, Tx_n) \leq k. \max\{d(x_n, y_n), d(y_n, Tx_n), d(x_n, Tx_n)\}.$$

So
$$d(Ty_n, Tx_n) \leq \frac{k}{1-k} \max\{d(x_n, y_n), d(x_n, Tx_n), d(y_n, Tx_n)\}.$$
Therefore, since $x_n \to x_0$, $Tx_n \to x_0$ and $y_n \to x_0$,
$$\begin{aligned}\lim_{n\to\infty} d(Ty_n, Tx_n) &\leq \frac{k}{1-k} \max\{d(x_0, x_0), d(x_0, x_0), d(x_0, x_0)\} = 0,\\ \text{and so } d(Ty_n, Tx_0) &\leq d(Ty_n, Tx_n) + d(Tx_n, x_0) \to 0.\end{aligned}$$

Hence $Ty_n \to Tx_0$ and so T is continuous at x_0.□

We next present a famous result of Kannan [Kan68].

Definition: [Kan68] Let (X, d) be a metric space. A mapping $T : X \to X$ is called a **Kannan mapping** if $\exists$ an $\beta \in (0, \frac{1}{2})$ such that
$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X.$$

Remark. (1) Contraction $\not\Leftrightarrow$ Kannan mapping; (2) Continuity $\not\Leftrightarrow$ Kannan mapping.

Example (1) Let $X = [0, 1]$ and $T : X \to X$ such that
$$Tx = \begin{cases} \frac{x}{4} \text{ if } 0 \leq x < \frac{1}{2} \\ \frac{x}{5} \text{ if } \frac{1}{2} \leq x \leq 1 \end{cases}$$

Then (i) T is not continuous (discontinuous at $x = \frac{1}{2}$) and consequently T is not contraction.

(ii) T is a Kannan mapping with $\beta = \frac{4}{9} < \frac{1}{2}$.

Note that T has a fixed point $x_0 = 0$.

Example (2) Let $X = [0, 1]$ and $T : X \to X$ such that $Tx = \frac{x}{3}$. Then

(i) T is a contraction with $\beta = \frac{1}{3} < 1$.

(ii) T is not a Kannan mapping.

Solution. (i) For any $x, y \in [0, 1]$,
$$d(Tx, Ty) = |Tx - Ty| = \left|\frac{x}{3} - \frac{y}{3}| = \frac{1}{3}|x - y\right| = \frac{1}{3}.d(x, y).$$

(ii) Take $x = \frac{1}{3}, y = 0$. Then
$$\begin{aligned}d(Tx, Ty) &= |Tx - Ty| = \left|T(\frac{1}{3}) - T(0)\right| = \left|\frac{1}{9} - 0\right| = \frac{1}{9},\\ d(x, Tx) + d(y, Ty) &= |x - Tx| + |y - Ty| = \left|\frac{1}{3} - \frac{1}{9}\right| + |0 - 0| = \frac{2}{9};\\ \text{so } d(Tx, Ty) &= \frac{1}{9} = \frac{1}{2}.\frac{2}{9} = \frac{1}{2}[d(x, Tx) + d(y, Ty)]\\ &\not< \frac{1}{2}[d(x, Tx) + d(y, Ty)].\end{aligned}$$

Hence T is not a Kannan mapping.

Theorem 2.3.2. [Kan68] *Let (X, d) be a complete metric space and $T : X \to X$ a kannan mapping. Then T has a unique fixed point.*

Proof. (I) Suppose $\exists$ an $\beta \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)] \text{ for all } x, y \in X. \tag{1}$$

Let x$\in X$. We want to show that $\{T^n x\}$ is a Cauchy sequence in X. Then for any $k \geq 1$,

$$\begin{aligned} d(T^k x, T^{k+1}x) &= d(T(T^{k-1}x), T(T^k x)) \\ &\leq \beta[d(T^{k-1}x, T^k x) + d(T^k x, T^{k+1}x)]; \\ \text{or, } d(T^k x, T^{k+1}x)[1-\beta] &\leq \beta d(T^{k-1}x, T^k x), \\ \text{or, } d(T^k x, T^{k+1}x) &\leq \frac{\beta}{1-\beta} d(T^{k-1}x, T^k x). \end{aligned} \tag{2}$$

Using (2) repeatedly, we obtain

$$d(T^k x, T^{k+1}x) \leq \left(\frac{\beta}{1-\beta}\right)^k d(x, Tx). \tag{3}$$

Now, if $m < n$, using (3)

$$\begin{aligned} d(T^m x, T^n x) &\leq d(T^m x, T^{m+1}x) + d(T^{m+1}x, T^{m+2}x) + \dots\dots + d(T^{n-1}x, T^n x) \\ &\leq \left(\frac{\beta}{1-\beta}\right)^m d(x, Tx) + \left(\frac{\beta}{1-\beta}\right)^{m+1} d(x, Tx) + \dots\dots + \left(\frac{\beta}{1-\beta}\right)^{n-1} d(x, Tx) \\ &= \left[\left(\frac{\beta}{1-\beta}\right)^m + \left(\frac{\beta}{1-\beta}\right)^{m+1} + \dots\dots + \left(\frac{\beta}{1-\beta}\right)^{n-1}\right] d(x, Tx) \\ &\leq \left[\left(\frac{\beta}{1-\beta}\right)^m + \left(\frac{\beta}{1-\beta}\right)^{m+1} + \dots\dots + \left(\frac{\beta}{1-\beta}\right)^{n-1} + \dots\dots\right] d(x, Tx) \\ &= \left[\frac{\left(\frac{\beta}{1-\beta}\right)^m}{1 - \frac{\beta}{1-\beta}}\right] d(x, Tx) \\ &= \left(\frac{\beta}{1-\beta}\right)^m \left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx). \end{aligned}$$

Since $0 < \beta < \frac{1}{2}$, we have $\frac{\beta}{1-\beta} < 1$ (since $2\beta < 1$) and $\lim_{m\to\infty} \left(\frac{\beta}{1-\beta}\right)^m = 0$. Then also

$$\lim_{m\to\infty} \left(\frac{\beta}{1-\beta}\right)^m \left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx) = 0.$$

So given $\varepsilon > 0$, $\exists$ an $N \geq 1$ such that

$$\left(\frac{\beta}{1-\beta}\right)^m \left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx) < \varepsilon \text{ for all } m \geq N.$$

Thus

$$d(T^m x, T^n x) \leq \left(\frac{\beta}{1-\beta}\right)^m \left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx) < \varepsilon \text{ for all } n > m \geq N.$$

Hence $\{T^n x\}$ is a Cauchy sequence in X.

(II) Since X is compelet, $T^n x \to x_0$ (say) in X. We want to show that $Tx_0 = x_0$. Now

$$\begin{aligned} d(x_0, Tx_0) &\leq d(x_0, T^n x) + d(T^n x, x_0) \\ &\leq d(x_0, T^n x) + \beta[d(T^{n-1}x, T^n x) + d(x_0, Tx_0)], \end{aligned}$$

or

$$\begin{aligned} d(x_0, Tx_0) &\leq \frac{1}{1-\beta} d(x_0, T^n x) + \frac{\beta}{1-\beta} d(T^{n-1}x, T^n x) \\ &\leq \frac{1}{1-\beta} d(x_0, T^n x) + \frac{\beta}{1-\beta}\left(\frac{\beta}{1-\beta}\right)^{n-1}\left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx) \\ &= \frac{1}{1-\beta} d(x_0, T^n x) + \left(\frac{\beta}{1-\beta}\right)^{n}\left(\frac{1-\beta}{1-2\beta}\right) d(x, Tx). \end{aligned}$$

Since $d(x_0, T^n x) \to 0$ and $\left(\frac{\beta}{1-\beta}\right)^n \to 0$, we have

$$d(x_0, Tx_0) = 0 \text{ or } Tx_0 = x_0.$$

(III) Uniqueness of x_0. Suppose also that $Ty_0 = y_0$ for some $y_0 \in X$. Then

$$d(x_0, y_0) = d(Tx_0, Ty_0) \leq \beta[d(x_0, Tx_0) + d(y_0, Ty_0)] = \beta[0 + 0] = 0,$$

hence $x_0 = y_0$. $\Box$

Next we present a more general result, due to Zamfirescu [Zam72], which includes Theorem 2.3.2 as a particular case.

Definition: [Zam72] Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called a **Zamfirescu operator** if there exist the real numbers α, β and γ satisfying $0 < \alpha < 1$, $0 < \beta, \gamma < 1/2$ such that for each pair $x, y \in X$, at least one of the following holds:

(z_1) (Banach) $d(T(x), T(y)) \leq \alpha d(x, y)$;

(z_2) (Kannan) $d(T(x), T(y)) \leq \beta[d(x, T(x)) + d(y, T(y))]$;

(z_3) (Chatterjea) $d(T(x), T(y)) \leq \gamma[d(x, T(y)) + d(y, T(x))]$.

Theorem 2.3.3. [Zam72] *Let (X, d) be a complete metric space and $T : X \to X$ a Zamfirescu operator. Then T has a unique fixed point.*

Proof. For any fixed $x_0 \in X$, consider the Picard iteration $\{T^n(x_0)\}_{n=0}^{\infty}$. We first show that $\{T^n(x_0)\}_{n=0}^{\infty}$ is a Cauchy sequence in X. Denoting

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

we have $0 < \delta < 1$.

If (z$_1$) holds, then

$$\begin{aligned} d(T^{n+1}(x_0), T^{n+2}(x_0)) &\leq \alpha d(T^n(x_0), T^{n+1}(x_0)) \\ &\leq \delta d(T^n(x_0), T^{n+1}(x_0)). \end{aligned}$$

If (z$_2$) holds, then

$$d(T^{n+1}(x_0), T^{n+2}(x_0)) \leq \beta[d(T^n(x_0), T^{n+1}(x_0)) + d(T^{n+1}(x_0), T^{n+2}(x_0))].$$

or

$$\begin{aligned} d(T^{n+1}(x_0), T^{n+2}(x_0)) &\leq \frac{\beta}{1-\beta} d(T^n((x_0)), T^{n+1}(x_0)) \\ &\leq \delta d(T^n(x_0), T^{n+1}(x_0)). \end{aligned}$$

If (z$_3$) holds, then

$$d(T^{n+1}(x_0), T^{n+2}(x_0)) \leq \gamma[d(T^n(x_0), T^{n+1}(x_0)) + d(T^{n+1}(x_0), T^{n+2}(x_0))].$$

or

$$\begin{aligned} d(T^{n+1}(x_0), T^{n+2}(x_0)) &\leq \frac{\gamma}{1-\gamma} d(T^n(x_0), T^{n+1}(x_0)) \\ &\leq \delta d(T^n(x_0), T^{n+1}(x_0)). \end{aligned}$$

Thus in each case,

$$\begin{aligned} d(T^{n+1}(x_0), T^{n+2}(x_0)) &\leq \delta d(T^n(x_0), T^{n+1}(x_0)) \\ &\leq \delta^2 d(T^{n-1}(x_0), T^n(x_0)) \\ &\leq \ldots\ldots. \\ &\leq \delta^n d((x_0), T((x_0))) \end{aligned}$$

Since $\delta^n \to 0$, it follows that $\{T^n(x_0)\}$ is a Cauchy sequence and therefore $T^n(x_0) \to z \in X$.

We now prove that z is a fixed point of T. Suppose $T(z) \neq z$ and consider the ball

$$B = \{x \in M : d(x, z) \leq \frac{1}{4} d(z, T(z))\}.$$

Observe that, for any $x \in B$,

$$\begin{aligned} d(z, T(z) &\leq d(z, x) + d(x, T(z) \leq \frac{1}{4} d(z, T(z) + d(x, T(z), \\ \text{so } \ d(x, T(z)) &> \frac{3}{4} d(z, T(z)). \end{aligned}$$

There exists a number N such that $T^n(x_0) \in B$ for each $n \geq N$. Taking now $x = T^N(x_0)$ and $y = z$, we must have again one of the next three situations:

(1)

$$d(T^{N+1}(x_0), T(z)) \leq \alpha d(T^N(x_0), z),$$

which, however, contradicts

$$\alpha d(T^N(x_0), z) \leq d(T^N(x_0), z) \leq \frac{1}{4} d(z, T(z)) < d(T^{N+1}(x_0), T(z));$$

(2)

$$d(T^{N+1}(x_0), T(z)) \leq \beta[d(T^N(x_0), T^{N+1}(x_0)) + d(z, T(z))]$$

contradicting

$$\begin{aligned} & \beta[d(T^N(x_0), T^{N+1}(x_0)) + d(z, T(z))] \\ < & \frac{1}{2}[d(T^N(x_0), z) + d(z, T^{N+1}(x_0)) + d(z, T(z))] \\ \leq & \frac{3}{4} d(z, T(z)) \leq d(T^{N+1}(x_0)), T(z)); \end{aligned}$$

(3)

$$d(T^{N+1}(x_0)), T(z)) \leq \gamma[d(T^N(x_0), T(z)) + d(T^{N+1}(x_0), z)]$$

respectively contradicting

$$\begin{aligned} & \gamma[d(T^N(x_0), T(z)) + d(T^{N+1}(x_0), z)] \\ < & \frac{1}{2}[d(T^N(x_0), z) + d(z, T(z)) + d(z, T^{N+1}(x_0))] \\ \leq & \frac{3}{4} d(z, T(z)) \leq d(T^{N+1}(x_0), T(z)). \end{aligned}$$

Thus, $T(z) = z$.

Now we show that this fixed point z is unique. Suppose this is not true: $T(z^*) = z^*$ for some point $z^* \in M$ different from z. Then

$$\begin{aligned} d(T(z), T(z^*)) &= d(z, z^*), \\ d(T(z), T(z^*)) &> d(z, T(z)) + d(z^*, T(z^*)), \\ td(T(z), T(z)) &= \frac{1}{2}[d(z, T(z^*)) + d(z^*, T(z))], \end{aligned}$$

so that none of the three conditions of the theorem is satisfied at the points z and z^*.

Corollary 2.3.4 (Banach). *Let X be a complete metric space, $\alpha < 1$, and $T : X \to X$ a function such that for each pair of different points in X condition (z_1) is verified. Then T has a unique fixed point.*

Corollary 2.3.5 (Kannan). *Let X be a complete metric space, $\beta < \frac{1}{2}$, and $T : X \to X$ a function such that for each pair of different points in M condition (z_2) is verified. Then T has a unique fixed point.*

Motivated by the Zamfirescu's results ([Zam72], Theorem 1), we consider the following.

Definition: [Os95, Be04] Let (X, d) be a complete metric space. A mapping $T : X \to X$ is called a $(\delta - L)$**-quasi-contractive mapping** with $0 \le \delta < 1$, $L > 0$ if it satisfies

$$d(T(x), T(y)) \le \delta.d(x, y) + L.d(x, T(x)) \text{ for all } x, y \in X; \qquad (DL)$$

or equivalently $d(T(x), T(y)) \le \delta.d(x, y) + L.d(y, T(y))$ for all $x, y \in X$.

Recall that a mapping $T : X \to X$ is called a **quasi-contractive** if it satisfies

$$d(T(x), u) < d(x, u) \text{ for all } x \in X \text{ and } u \in Fix(T). \qquad (Q)$$

Lemma 2.3.6 (a) *Every Zamfirescu operator T on X is $(\delta - L)$-quasi-contractive for some $0 \le \delta < 1$ and $L = 2\delta$.*

(b) *Every $(\delta - L)$-quasi-contractive mapping T on X is quasi-contractive.*

(c) *Every $(\delta - L)$-quasi-contractive mapping T on X has at most one fixed point.*

Proof. (a) This follows essentially from the proof of Theorem 1 of ([Zam72], theorem 1). In fact, let α, β and γ be the numbers with $0 < \alpha < 1$, $0 < \beta, \gamma < 1/2$ satisfying (z_1), (z_2) and (z_3). If (z_2) holds, then

$$\begin{aligned} d(Tx, Ty) &\le \beta[d(x, Tx) + d(y, Ty)] \\ &\le \beta[d(x, Tx) + d(y, x) + d(x, Tx) + d(Tx, Ty)] \\ &\le 2\beta d(x, Tx) + \beta d(y, x) + \beta d(Tx, Ty), \end{aligned}$$

and so

$$d(Tx, Ty) \le \frac{\beta}{1-\beta} d(x, y) + \frac{2\beta}{1-\beta} d(x, Tx). \qquad (1)$$

Similarly, if (z_3) holds, then

$$d(Tx, Ty) \le \frac{\gamma}{1-\gamma} d(x, y) + \frac{2\gamma}{1-\gamma} d(x, Tx). \qquad (2)$$

Denoting

$$\delta = \max\{\alpha, \frac{\beta}{1-\beta}, \frac{\gamma}{1-\gamma}\},$$

we have $0 < \delta < 1$ and by (z_1), (1) and (2),

$$d(Tx, Ty) \le \delta d(x, y) + 2\delta d(x, Tx) \text{ for all } x, y \in K.$$

Thus T is a $(\delta - L)$-quasi-contractive mapping with $L = 2\delta$.

(b) Suppose T is $(\delta{-}L)$-quasi-contractive. For $x \in X$ and $u \in Fix(T)$, by (DL), we obtain

$$d(Tx,u) = d(Tx,Tu) \leq \delta.d(x,u) + L.d(u,Tu) = \delta.d(x,u) + L.0 < d(x,u).$$

Hence T is quasi-contractive.

(c) Let $u, w \in Fix(T)$ with $u \neq w$. Then, by (DL),

$$\begin{aligned} d(u,w) &= d(Tu,Tw) \leq \delta d(u,w) + 2\delta d(u,Tu) \\ &= \delta d(u,w) + 2\delta.0 = \delta d(u,w) < d(u,w), \end{aligned}$$

a contradiction. Hence $u = w$. □

We next consider convergence of Mann Iterates to fixed points in Banach spaces

Definition: Let E be a Banach space, K a convex subset of E and $T : K \to K$ a given mapping. For any $x_0 \in K$ and $\{\alpha_n\} \subseteq [0,1]$, the sequence $\{x_n\} \subseteq K$ defined by

$$x_{n+1} = (1-\alpha_n)x_n + \alpha_n T x_n, \qquad n = 0, 1, 2, ... \tag{M}$$

is called the **Mann iteration** (first introduced by Mann in1953).

Particular Cases: (1) For $\alpha_n = \lambda$ (constant), the iteration (M) reduces to

$$x_{n+1} = (1-\lambda)x_n + \lambda T x_n, n = 0, 1, 2, ... \tag{Kr}$$

the so called **Krasnoselskij iteration**.

(2) For $\alpha_n = 1$ in (M) we obtain

$$x_{n+1} = T(x_n), n = 0, 1, 2, ... \tag{P}$$

the **Picard iteration** or **method of successive approximations** (see Berinde's Book (2007)).

Theorem 2.3.7. [Ber04] *Let $E = (E, ||.||)$ be a Banach space, K a closed convex subset of E, and $T : K \to K$ a $(\delta - L)$-quasi-contractive mapping. Suppose $Fix(T) \neq \varnothing$. Let $\{x_n\}_{n=0}^{\infty}$ be the Mann iteration defined by (M) and $x_0 \in K$, where $\{\alpha_n\}$ is the sequence of positive numbers in $[0,1]$ satisfying*

$$\sum_{n=0}^{\infty} \alpha_n = \infty, \textit{ or equivalently } \prod_{n=0}^{\infty} (1-\alpha_n) = 0 \text{ ([Deu01], p. 145-146)} \tag{A}$$

Then $\{x_n\}_{n=0}^{\infty}$ converges strongly to the fixed point of T.

Proof. By hypothesis, T has a unique fixed point in K, say u. Also by hypothesis, there exist $0 \leq \delta < 1$ and $L \geq 0$ such that

$$||Tx - Ty|| \leq \delta||x - y|| + L||x - Tx|| \text{ for all } x, y \in X. \tag{DL}$$

Now let $\{x_n\}$ be the Mann iteration defined by (M) and $x_0 \in K$ arbitrary. Then

$$\begin{aligned} ||x_{n+1} - u|| &= ||(1-\alpha_n)x_n + \alpha_n T x_n - (1-\alpha_n+\alpha_n)u|| \\ &= ||(1-\alpha_n)(x_n - u) + \alpha_n(Tx_n - u)|| \\ &\leq (1-\alpha_n)||x_n - u|| + \alpha_n||Tx_n - u||. \end{aligned} \tag{1}$$

With $x = u$ and $y = x_n$, from (DL) we obtain

$$||Tx_n - u|| \leq \delta||x_n - u||. \tag{2}$$

By putting (2) in (1) we obtain

$$\begin{aligned} ||x_{n+1} - u|| &\leq (1-\alpha_n)||x_n - u|| + \alpha_n||Tx_n - u||, \text{ by (1)} \\ &\leq (1-\alpha_n)||x_n - u|| + \alpha_n.||x_n - u||, \text{ by (2)} \\ &= [1-(1-\delta)\alpha_n].||x_n - u|| \\ &= \gamma_n.||x_n - u||, \quad \gamma_n = 1-(1-\delta)\alpha_n. \end{aligned} \tag{3}$$

Continuing this process, we obtain

$$\begin{aligned} ||x_{n+1} - u|| &\leq \gamma_n.\gamma_{n-1}.||x_{n-1} - u|| \\ &\leq \ldots\ldots\ldots \\ &\leq \gamma_n.\gamma_{n-1}....\gamma_1.\gamma_0.||x_0 - u|| \\ &= \prod_{k=0}^{n} \gamma_k.||x_0 - u||]. \end{aligned} \tag{4}$$

for all $n = 0, 1, 2, ...$Using the fact that $0 \leq \delta < 1$ and $\alpha_k, \beta_n \in [0,1]$ satisfy (A), it follows that

$$\lim_{n\to\infty} \prod_{k=0}^{n} \gamma_n = \lim_{n\to\infty} \prod_{k=0}^{n} [1-(1-\delta)\alpha_k] = 0.$$

Consequently, by (4),

$$\lim_{n\to\infty} ||x_{n+1} - u|| = 0.$$

Thus $\{x_n\}$ converges strongly to u, a fixed point of T. $\square$

Remarks. (1) In the setting of metric linear spaces, the results of this section are in contrast to those in [Khan88, Khan92, Rh77], where assuming $\{x_n\}$ to be strongly convergent to $u \in K$, it is shown that u is fixed point of T. While here (in the setting of normed spaces), we assume that T has a fixed point $u \in K$ and we show that $\{x_n\}$ is strongly convergent to u.

2.4 Fixed Point Theorems for Multivalued Contraction Mappings

Banach's Contraction Mappings Principle was extended to set-valued mappings by S. Nadler [Nad69]. In this section we present fixed point theorem for contraction multivalued mappings.

Let (X, d) be a metric space. Recall from Section 1.4 that 2^X is the family of all non-empty subsets of X and $b(X)$ (resp., $cb(X), k(X)$) denote the family of all non-empty bounded (resp., closed and bounded, compact) subsets of X. Clearly,

$$k(X) \subseteq cb(X) \subseteq b(X) \subseteq 2^X.$$

Definition: A function $T : X \to 2^X$ is called a **multivalued** (or s**et-valued**) mapping. A point $x_0 \in X$ is called a f**ixed point** of T if $x_0 \in T(x_0)$. We define $Fix(T) = \{x \in X : x \in T(x)\}$.

Clearly, if T is single valued, i.e. if $T : X \to X$, then $x_0 \in T(x_0)$ means $x_0 = T(x_0)$.

Example. Let $X = [0, \infty)$ with usual metric. Define $T : X \to 2^X$ by

$$T(x) = \begin{cases} [0, x] & \text{if} \quad x \in [0, 1) \\ \{0\} & \text{if} \quad x \geq 1. \end{cases} \quad , x \in X.$$

Then $Fix(T) = [0, 1)$.

Solution. If $x \in [0, 1)$, then $T(x) = [0, x]$, so clearly $x \in [0, x] = T(x)$. If $x \geq 1$, then $T(x) = \{0\}$, and so $x \notin \{0\} = T(x)$. Thus $Fix(T) = [0, 1)$. $\square$

Definition: Let (X, d_1) and (Y, d_2) be metric spaces. A function $T : X \to cb(Y)$ is said to be a **multi-valued contraction mapping** of X into Y if there exists a number $\alpha, 0 \leq \alpha < 1$ such that

$$d_H(T(x), T(y)) \leq \alpha d_1(x, y) \text{ for all } x, y \in X.$$

Theorem 2.4.1. [Nad69] *Let (X, d) be a complete metric space. If $T : X \to cb(X)$ is a multi-valued contraction mapping, then T has a fixed point.*

Proof. (I) Since T is a contraction, $\exists$ an $\alpha \in [0, 1)$ such that

$$d_H(T(x), T(y)) \leq \alpha d(x, y) \text{ for all } x, y \in X. \tag{1}$$

By definition of d_H, given any $A, B \in cb(X)$ and $a \in A$, for each $\varepsilon > 0$, $\exists$ $b = b_\varepsilon \in B$ such that

$$\begin{aligned} d(a, b) &\leq d(a, B) + \varepsilon \quad (\text{since } d(a, B) = \inf_{b \in B} d(a, b)) \\ &\leq \sup_{a' \in A} d(a', B) + \varepsilon \\ &= d(A, B) + \varepsilon \\ &\leq d_H(A, B) + \varepsilon. \end{aligned} \tag{2}$$

(II) Let $x_0 \in X$. Then $T(x_0) \in cb(X)$. Choose any $x_1 \in T(x_0)$. Now using (2) with $A = T(x_0), B = T(x_1), x_1 \in A$ and $\varepsilon = \alpha$, $\exists\ x_2 \in B = T(x_1)$ such that

$$d(x_1, x_2) \leq d_H(T(x_0), T(x_1)) + \alpha.$$

Next, take $A = T(x_1), B = T(x_2), x_2 \in A$ and $\varepsilon = \alpha^2$,by (2), $\exists\ x_3 \in B = T(x_2)$ such that

$$d(x_2, x_3) \leq d_H(T(x_1), T(x_2)) + \alpha^2.$$

Continuing this process, we get $\{x_n\}$ with $x_{n+1} \in T(x_n)$ such that

$$d(x_n, x_{n+1}) \leq d_H(T(x_{n-1}), T(x_n)) + \alpha^n. \tag{3}$$

(III) $\{x_n\}$ is Cauchy in (X, d).

For any integer $k \geq 1$,

$$\begin{aligned}
d(x_k, x_{k+1}) &\leq d_H(T(x_{k-1}), T(x_k)) + \alpha^k \quad \text{[using (3)]}\\
&\leq \alpha d(x_{k-1}, x_k) + \alpha^k \quad \text{[by (1)]}\\
&\leq \alpha[d_H(T(x_{k-2}), T(x_{k-1})) + \alpha^{k-1}] + \alpha^k \quad \text{[by (3)]}\\
&\leq \alpha[\alpha d(x_{k-2}, x_{k-1}) + \alpha^{k-1}] + \alpha^k \quad \text{[by (1)]}\\
&= \alpha^2 d(x_{k-2}, x_{k-1}) + 2\alpha^k\\
&\leq \ldots\ldots\ldots\ldots\\
&\leq \alpha^k d(x_0, x_1) + k.\alpha^k.
\end{aligned} \tag{4}$$

Taking $\sum\limits_{k=0}^{\infty}$ for both sides of (4), we get

$$\begin{aligned}
\sum_{k=0}^{\infty} d(x_k, x_{k+1}) &\leq \sum_{k=0}^{\infty} \alpha^k d(x_0, x_1) + \sum_{k=0}^{\infty} k.\alpha^k\\
&< \infty \quad (\text{since } 0 \leq \alpha < 1).
\end{aligned}$$

Hence $d(x_k, x_{k+1}) < \infty\ \forall\ k \geq 1$.So by Cauchy criterion, $\{x_n\}$ is a a Cauchy sequence.

(IV) Since (X, d) is complete, the sequence $\{x_n\}$ converges to z (say) in X. Therefore, by continuity of T,the sequence $\{T(x_n)\}$ converges to $T(z)$ and,by continuity of d_H,

$$\lim_{n\to\infty} d_H(T(x_n), T(z)) = d_H(T(z), T(z)) = 0.$$

i.e,

$$\lim_{n\to\infty} \max\{d(T(x_n), T(z)), d(T(z), T(x_n))\} = 0$$

Then

$$
\begin{aligned}
&\Rightarrow \quad \lim_{n\to\infty} d(T(x_n), T(z)) = 0 \\
&\Longrightarrow \quad \lim_{n\to\infty} d(x_{n+1}, T(z)) = 0 \quad (\text{since } x_{n+1} \in T(x_n)) \\
&\Longrightarrow \quad d(z, T(z)) = 0 \\
&\Longrightarrow \quad z \in \overline{T(z)} = T(z) \quad (\text{since } T(z) \in cb(X)).
\end{aligned}
$$

Thus z is a fixed point of T. □

2.5 Retracts in Fixed Point Theory

Recall that: if X is a topological space and $A \subseteq X$, then A is called a **retract** of X if there exists a continuous function $r_A : X \to A$ such that $r_A(x) = x$ for all $x \in A$. r_A is called a **retraction** of X into A. If X is Hausdorff, and $A \subseteq X$ a retract of X, then A is closed in X. (See Section 1.1; ([KK01], p. 176.)

The concept of a retract is directly related to the problem of the extension of continuous mappings.

Theorem 2.5.1. (Characterization of Retract) *A subspace A is a retract of topological space X iff every continuous mapping of A into an arbitrary topological space Y can be extended to a continuous mapping of the entire space X into Y.*

As examples:

(1) Any topological space X is a retract of X (under the identity mapping).

(2) For each $x_0 \in X, \{x_0\}$ is a retract of X (under the constant mapping $r : X \to \{x_0\}$ such that $r(x) = x_0$ for all $x \in \{x_0\}$.

(3) If $X = \mathbb{R}^n$ and $A = B[0,1] = \{x \in \mathbb{R}^n : \|x\| \leq 1\}$, the closed unit ball in $\mathbb{R}^n$, then A is a retract of X.

(4) If $X = [a,b]$, a closed subspace of $\mathbb{R}$ and $A = \{a,b\}$, then A is cannot be a retract of X.

Definition: Let (X,τ) be a topological space and $C \subseteq X$. Then C is said to have the **fixed-point property** (in short, **FPP)** if every continuous map $f : C \to C$ (not necessarily a self-homeomorphism) has a fixed point.

Theorem 2.5.2. *Suppose a topological space X has the FPP. Then any retract A of X has also the FPP.*

Proof. Consider a retraction $r : X \to A$ where $A \subseteq X$ and $r(a) = a$ for all $a \in A$. Let $i : A \to X$ denote the inclusion of A in X.

Consider any continuous map $f : A \to A$. We need to show that f has a fixed point in A. Consider the composition $g = i \circ f \circ r : X \to X$. This is a map that first retracts X to A , then applies f, and then views the resulting point of A as a point in X. g is a composite of continuous maps, so g is continuous. Since X has the FPP, there exists $x \in X$ such that $g(x) = x$.

But by construction, clearly $g = i \circ f \circ r : X \to A$, and so $g(x) \in A$, hence $x \in A$. But if $x \in A$, $r(x) = x$, so we conclude that $x = g(x) = f(r(x)) = f(x)$. Thus, $x \in A$ is a fixed point of f. Thus A has the FPP. □

Remarks. (1) The FPP is a topological invariant, i.e. is preserved by any homeomorphism.

(2) The FPP is also preserved by any retraction.

(3) According to Brouwer fixed point theorem every compact and convex subset of an Euclidean space has the FPP.

Next we consider "nonexpansive retracts" which are of interest in fixed point theory.

Definition: Let X be a real Banach space, C a nonempty closed convex subset of X, and $K \subseteq C$ a nonempty closed set. We do not assume that K is convex. Then K is said to be a **nonexpansive retract** of C if either $K = \varnothing$ or there exists a retraction r_K of C onto K which is a nonexpansive mapping.

The following result is a well-known (Section 2.1):

Theorem 2.5.3. If X is a strictly convex Banach space, C a nonempty closed convex subset of X and $T : C \to C$ is nonexpansive, then $Fix(T)$ is a closed convex subset of C.

Recall that: For each $x \in X$, let

$$P_C(x) := \{z \in C \,:\, d(x, z) = d(x, C)\}.$$

Then each $z \in C$ is called a **point of best approximation** of C from x. The map $P_C : X \to C$ is called a **proximity map**.

The classical result about retraction is:

Theorem 2.5.4. ([KK01], p. 135-136) *Let $(X, <, >)$ be a Hilbert space and C a nonempty closed convex subset of X. Then the proximity mapping $P_C : X \to C$ is nonexpansive.*

Proof. See Section 1.7.

Theorem 2.5.5. *Let X be a Hilbert space, C a nonempty closed convex subset of X and $T : C \to C$ a nonexpansive map. Then the proximity mapping $P_C : X \to C$ is nonexpansive, hence also continuous. Consequently, C is a nonexpansive retract of X.*

Remarks. (1) [Bru73b] If X is a Hilbert space and $T : C \to C$ is nonexpansive, then $Fix(T)$ is a closed convex subset of C (Section 2.1). If $Fix(T)$ is nonempty, the proximity mapping $P_C : C \to Fix(T)$ defined by $P_C(x) =$ the point of $Fix(T)$ which is closest to x, is nonexpansive. Consequently, $Fix(T)$ is a nonexpansive retract of C.

(2) [KopRe07] Let C be a nonexpansive retract of the Banach space X. If $T : C \to Y$ is a Lipschitz mapping, Y a metric space, then T can be extended to all of X without increasing its Lipschitz constant.

(3) ([KK01], p. 176-177) Every closed convex subset K of $\mathbb{R}^n$ is a retract of $\mathbb{R}^n$.

(5) ([KK01], p. 177) The surface $S = \{x \in \mathbb{R}^n : ||x|| = r\}$ of a nontrivial closed ball $B[0,r]$ of $\mathbb{R}^n$ is not a retract of $\mathbb{R}^n$.

Theorem 2.5.6. (Bruck [Bru72]) *Let X be a strictly convex reflexive Banach space, C a nonempty closed convex subset of X, $K \subseteq C$, and $T : C \to C$ is nonexpansive. Then $Fix(T)$ is a nonexpansive retract of C.*

Proof. Suppose K is a nonempty subset of C, and let

$$\mathcal{F} = \{T : C \to C, T \text{ is nonexpamsive and } K \subseteq Fix(T)\}$$

Define an order on $\mathcal{F}$ by setting

$$f < g \text{ if } ||f(x) - f(y)|| \leq ||g(x) - g(y)|| \text{ for all } (x,y) \in C \times C,$$

with strict inequality holding for at least one pair (x,y); then let $f \leq g$ to mean $f < g$ or $f = g$. Then $\leq$ is a partial ordering of $\mathcal{F}$

Every linearly ordered subset of $\mathcal{F}$ has a lower bound in $\mathcal{F}$; the proof of this fact utilizes the local weak compactness of C and the weak lower semicontinuity of the norm. Therefore, by Zorn's lemma, $\mathcal{F}$ has a minimal element.

The strict convexity of X implies that for each $g \in \mathcal{F}$ there exists a $g_0 \in \mathcal{F}$ with $Fix(g_0) = Fix(g)$ and such that whenever

$$||g_0(u) - g_0(w)|| = ||u - w||,$$

then $g_0(u) - g_0(w) = u - w$. For example, we may take $g_0 = \frac{1}{2}I + \frac{1}{2}g_0$, where I is the identity function for C.

Suppose f is a minimal function in $\mathcal{F}$, and g is any function of $\mathcal{F}$ Let g_0 be the function of the preceding paragraph; then $g_0 f \in \mathcal{F}$ while $g_0 f \leq f$. By the minimality of f, therefore $g_0 f \leq f$. Further,

(1) $Fix(f) \subseteq f(C) \subseteq Fix(g_0) = Fix(g)$,

and in particular,

(2) $Fix(f) \subseteq Fix(g)$ for all $g \in \mathcal{F}$.

Taking $g = f$ in (1), we see that $Fix(f) = f(C)$, so that f is a nonexpansive retraction onto $Fix(f)$. From (2), if f and g are minimal elements of $\mathcal{F}$, then $Fix(f) = Fix(g)$.

We claim that this common set $Fix(f)$ is the smallest nonexpansive retract K_1 of C with $K \subseteq K_1$. If the g of (2) is any nonexpansive retraction with $Fix(g) = K_1$, we have from (2) that $K \subseteq Fix(f) \subseteq K_1$. Thus $Fix(f)$ is the smallest nonexpansive retract K_1 of C with $K \subseteq K_1$.

Now suppose $Fix(T) \neq \varnothing$. Set $K = Fix(T)$ and let f be a minimal element of $\mathcal{F}$ Taking $g = T$ in (2), $Fix(f) \subseteq Fix(T)$, while $Fix(T) \subseteq Fix(f)$ in order for $f \in \mathcal{F}$; thus f is a nonexpansive retraction of C onto $Fix(f) = Fix(T)$. $\square$

Other related concept are:

Definition: [Bor67; DG82] (i) A subspace A of a topological space X is called a **neighbourhood retract** of X if there is in X an open subspace which contains A and of which A is a retract.

(ii) A metric space X is called an **absolute retract** (in short **AR**) if it is a retract (neighbourhood retract) of every metric space containing X as a closed subspace.

(iii) A metric space X is called an **absolute neighbourhood retract** (in short, **ANR**) if it is a neighbourhood retract of every metric space containing X as a closed subspace.

(iv) A **homotopy** between two continuous functions $f, g : X \to Y$ (X, Y topological spaces) is defined to be a continuous function $H : X \times [0, 1] \to Y$ such that

$$H(x, 0) = f(x) \text{ and } H(x, 1) = g(x) \quad \text{for all } x \in X.$$

(v) A topological space X is **contractible** if the identity map on X is homotopic to some constant map. Intuitively, a contractible space is one that can be continuously shrunk to a point.

Theorem 2.5.7 (Characterization of AR) *For a metric space X to be an AR it is necessary that it be a retract of some convex subspace of a normed vector space, and it is sufficient that X be a retract of a convex subspace of a locally convex TVS.*

Remarks (Properties of AR and ANR).The above characterization means that ARs have the following properties.

(1) All convex subspaces of locally convex linear spaces are ARs. (Such is the case, in particular, with a point, an interval, a ball, a straight line, etc.)

(2) Every retract of an AR is again an AR.

(3) Each AR is contractible in itself and is locally contractible.

(4) A metric space Y is an AR iff, given any metric space X, a closed subspace A of X and a continuous mapping of A into Y, the mapping can be extended to a continuous mapping of the entire space X into Y.

(5) ANRs are characterized as retracts of open subsets of convex subspaces of normed vector spaces. They include all compact polyhedra.

Theorem 2.5.8 [DG82] (a) (Arens-Eells) *Any metric space Y can be isometrically embedded as a closed subset in a normed vector space.*

(b) *A metrizable space Y is an AR iff it is a retract of every metrizable space in which it is embedded as a closed set.*

Chapter 3

W-Convex Metric Spaces and their Applications

In this chapter we study W-convex metric spaces (in short, WCM spaces) and their various properties. These include strict and uniform convexity. Further, we consider its applications in Fixed Point Theory, Best approximation and Invariant Approximation. Also we include the convergence of certain iterations of mappings to their fixed points..

3.1 W-Convex Metric Space and Its Properties

This section contains the definition of W-convex metric space and its various properties. This notion of convexity was first introduced by Takahashi in 1970 [Tak70], who also generalize some fixed point theorems of Banach spaces. Subsequently, Machado (1973), Talman (1977), and Naimpally, Singh and Whitfield (1983) and Beg and Azam (1986), among others, developed this theory and obtained fixed point theorems in W-convex metric spaces.

Definition: Let (X, d) be a metric space and $I = [0, 1]$. A function $W : X \times X \times I \longrightarrow X$ is called a W-**convex structure** if, for any $(x, y, t) \in X \times X \times I$ and $z \in X$,

$$d(z, W(x, y, t)) \leq td(z, x) + (1 - t)d(z, y) \text{ for all } z \in X. \tag{W}$$

The set X with this W-convex structure$W$ is called a W-**convex metric space** (in short, a **WCM space**) and is denoted by (X, d, W).

Definition: (a) (X, d) is called W'-**convex** if there exists a map $W : X \times X \times [0, 1] \to X$ such that for any $x, y \in X$ and $t \in [0, 1]$, we have

$$d(x, W(x, y, t)) + d(y, W(x, y, t)) = d(x, y). \tag{W'}$$

(b) (X, d) is called W^***-convex** if there exists a map $W : X \times X \times [0, 1] \to X$ such that for any $x, y \in X$ and $t \in [0, 1]$, we have

$$d(x, W(x, y, t)) = (1 - t)d(x, y) \text{ and } d(y, W(x, y, t)) = td(x, y). \qquad (W^*)$$

Theorem 3.1.1. $(W) \Rightarrow (W^*) \Rightarrow (W')$.

Proof. $(W) \Rightarrow (W^*)$. First notice that (by taking $z = x$ in $(W\ )$)

$$d(x, W(x, y, t)) \le (1 - t)d(x, y) \qquad \text{(I)}$$

and (by taking $z = y$ in $(W\ )$)

$$d(y, W(x, y, t)) \le td(x, y). \qquad \text{(II)}$$

Then

$$\begin{aligned} d(x, y) &\le d(x, W(x, y, t)) + d(W(x, y, t), y) \\ &\le d(x, W(x, y, t)) + td(x, y)] \quad \text{(by } (I)), \end{aligned}$$

or, $(1 - t)d(x, y) \le d(x, W(x, y, t))$; hence $d(x, W(x, y, t)) = (1 - t)d(x, y)$. Next,

$$\begin{aligned} d(x, y) &\le d(x, W(x, y, t)) + d(W(x, y, t), y) \\ &\le (1 - t)d(x, y) + d(y, W(x, y, t)), \text{ by } (II), \end{aligned}$$

or $td(x, y) \le d(y, W(x, y, t))$; hence $d(y, W(x, y, t)) = td(x, y)$. Thus $(W) \Rightarrow (W^*)$.

$(W^*) \Rightarrow (W')$. By adding the two equations in (W^*), we obtain

$$d(x, W(x, y, t)) + d(y, W(x, y, t)) = td(x, x) + (1 - t)d(x, y) = d(x, y). \square$$

Question. Does $(W') \Rightarrow (W^*)$?

Can we modify Example (1) (below) to show that $(W') \not\Rightarrow (W^*)$?

We now give two examples on a WCM space.

Example 1. Let K be a convex subset of Banach space $(X, \|.\|)$. Then K is a WCM space.

Solution. Define $W : K \times K \times I \longrightarrow K$ by

$$W(x, y, t) = tx + (1 - t)y, \ \forall x, y \in K, t \in I.$$

Then W is convex structure on K. So (K, W, d) is a WCM space.

Solution. To show that W is a convex structure, let $z \in X$.

$$\begin{aligned} d(z, W(x, y, t)) &= d(z, tx + (1 - t)y) \\ &= \|z - [tx + (1 - t)y]\| \\ &= \|tz + (1 - t)z - [tx + (1 - t)y]\| \\ &= \|t(z - x) + (1 - t)(z - y)\| \\ &\le t\|z - x\| + (1 - t)\|z - y\| \\ &= td(z, x) + (1 - t)d(z, y). \square \end{aligned}$$

Remark. Any Banach space is WCM space.

Example 2. Let (X, d) be a metric linear space with d a translation invariant metric:

$$d(x, y) = d(x - y, 0) = d(x + z, y + z) \text{ for all } x, y, z \in X.$$

Suppose also that d also satisfies:

$$d(tx + (1 - t)y, 0) \leq td(x, 0) + (1 - t)d(y, 0) \text{ for all } x, y \in X, 0 \leq t \leq 1.$$

Define $W : X \times X \times I \longrightarrow X$ by

$$W(x, y, t) = tx + (1 - t)y, \text{ for all } x, y \in X, t \in I.$$

Then W is a W-convex structure.

Solution.

$$\begin{aligned} d(z, W(x, y, t)) &= d(z, tx + (1 - t)y) \\ &= d(z - [tx + (1 - t)y], 0) \\ &= d(tz + (1 - t)z - [tx + (1 - t)y], 0) \\ &= d(t(z - x) + (1 - t)(z - y), 0) \\ &\leq td(z - x, 0) + (1 - t)d(z - y, 0) \\ &= td(z, x) + (1 - t)d(z, y). \end{aligned}$$

Definition: Let (X, d, W) be a WCM space. A subset K of X is called **convex** *or* **(W-convex)** if

$$W(x, y, t) \in K \text{ for any } x, y \in K, 0 \leq t \leq 1$$

Note. In general, $W(x, y, t) \neq tx + (1 - t)y$, since X need not be a vector space.

Theorem 3.1.2. **(**[Tak70], p. 143**)** *Let (X, d, W) be a WCM space. Then*

(a) *If $\{K_\alpha : \alpha \in A\}$ is a family of W-convex subsets of X, then $\cap_{\alpha \in A} K_\alpha$ is also W-convex.*

(b) *The open and closed balls $B(x, r)$, $B[x, r]$ respectively are W-convex subsets of X.*

(c) *For any $x, y \in X$ and $0 \leq t \leq 1$,*

$$d(x, y) = d(x, W(x, y, t)) + d(W(x, y, t), y) \qquad (W')$$

Proof. (a) Let $x, y \in \cap_{\alpha \in A} K_\alpha$ and $0 \leq t \leq 1$. Then

$$\begin{aligned} &\Longrightarrow \quad x, y \in K_\alpha, \forall \alpha \in A, \\ &\Longrightarrow \quad W(x, y, t) \in K_\alpha, \forall \alpha \in A, \quad \text{(since each } K_\alpha \text{ is } W\text{-convex)} \\ &\Longrightarrow \quad W(x, y, t) \in \cap_{\alpha \in A} K_\alpha. \end{aligned}$$

Hence, $\cap_{\alpha\in A}K_\alpha$ is W-convex.

(b) $B(x,r)$ is W-convex: Let $u,v\in B(x,r)$, $0\le t\le 1$.Then

$$\begin{aligned} d(W(u,v,t),x) &= d(x,W(u,v,t)) \\ &\le td(x,u)+(1-t)d(x,v) \\ &< tr+(1-t)r=r \end{aligned}$$

Hence, $W(u,v,t)\in B(x,r)$. So $B(x,r)$ is$W$-convex.

Similarly, $B[x,r]$ is W-convex.

(c)

$$\begin{aligned} d(x,y) &\le d(x,W(x,y,t))+d(W(x,y,t),y) \\ &\le td(x,x)+(1-t)d(x,y)+td(y,x)+(1-t)d(y,y), \quad \text{(by } (W)) \\ &= d(x,y)[(1-t)+t]=d(x,y). \end{aligned}$$

Hence, $d(x,y)=d(x,W(x,y,t))+d(W(x,y,t),y)$. □

Theorem 3.1.3. [Tak70, Mach73, Tal77] *Let* (X,d,W) *be a* WCM *space. Then for any* $x,y\in X$ *and* $t,s\in[0,1]$, *the following hold.*

(a) (i) $W(x,x,t)=x$, (ii) $W(x,y,0)=y$, (iii) $W(x,y,1)=x$.

(b) (i) $d(x,W(x,y,t))=(1-t)d(x,y)$.

(ii) $d(y,W(x,y,t))=td(x,y)$.

(c) $d(x,y)=d(x,W(x,y,t))+d(W(x,y,t),y)$ $\qquad (W')$

(d) $|t-s|\,d(x,y)\le d(W(x,y,t),W(x,y,s))$.

Proof: (a) (i) To show:$W(x,x,t)=x$, put $z=x$ and $y=x$ in (W).Then

$$d(x,W(x,x,t))\le td(x,x)+(1-t)d(x,x)=0.$$

Hence, $d(x,W(x,x,t))=0$, or $W(x,x,t)=x$.

(a) (ii) To show :$W(x,y,0)=y$, put $z=y$ and $t=0$ in (W).Then

$$d(y,W(x,y,0))\le 0.d(y,x)+1.d(y,y)=0$$

Hence, $d(y,W(x,y,0))=0$ or $W(x,y,0)=y$.

(a) (iii) To show: $W(x,y,1)=x$, put $z=x$ and $t=1$ in (W).Then

$$d(x,W(x,y,1))\le 1.d(x,x)+(1-1)d(x,y)=0.$$

Hence, $d(x,W(x,y,1))=0$, or$W(x,y,1)=x$.

(b) (i) To Show: $d(x,W(x,y,t))=(1-t)d(x,y)$.

First: we want to show that: $d(x,W(x,y,t))\le(1-t)d(x,y)$.Putting $z=x$ in (W),

$$d(x,W(x,y,t))\le td(x,x)+(1-t)d(x,y)=(1-t)d(x,y). \tag{1}$$

Next,

$$\begin{aligned} d(x,y) &\leq d(x,W(x,y,t))+d(W(x,y,t),y) \\ &\leq d(x,W(x,y,t))+[td(x,y)+(1-t)d(y,y)] \quad \text{(by } (W)), \\ \text{or } (1-t)d(x,y) &\leq d(x,W(x,y,t)). \qquad (2) \end{aligned}$$

By (1) and (2), $d(x,W(x,y,t))=(1-t)d(x,y)$

(b) (ii) First: we want To show that: $d(y,W(x,y,t))\leq td(x,y)$.Putting $z=y$ in (W),

$$d(y,W(x,y,t))\leq td(y,x)+(1-t)d(y,y)=td(x,y). \qquad (3)$$

Next,

$$\begin{aligned} d(x,y) &\leq d(x,W(x,y,t))+d(W(x,y,t),y) \quad \text{(by (T-Ineq))} \\ &\leq [td(x,x)+(1-t)d(x,y)]+d(W(x,y,t),y) \quad \text{(by } (W)), \\ \text{or } td(x,y) &\leq d(W(x,y,t),y). \qquad (4) \end{aligned}$$

By (3) and (4), $d(y,W,(x,y,t))=td(x,y)$.

(c) As already proved in (W'), $d(x,y)=d(x,W(x,y,t))+d(W(x,y,t),y)$.

(d)

$$\begin{aligned} |t-s|\,d(x,y) &= |(t-s)d(x,y)|=|td(x,y)-sd(x,y)| \\ &= |d(y,W(x,y,t))-d(y,W(x,y,s))|\,, \quad \text{(by (b)(ii))} \\ &\leq d(W(x,y,t),d(W(x,y,s)). \end{aligned}$$

Hence, $|t-s|\,d(x,y)\leq d(W(x,y,t),W(x,y,s))$. □

Theorem 3.1.4. ([Tal77], p. 64) *Let (X,d,W) be a WCM space. Then W is a continuous at each point (x,x,t) of $X\times X\times I$.*

Proof. Let $\{(x_n,y_n,t_n)\}_{n=1}^{\infty}$ be a sequence in $X\times X\times I$ which converges to (x,x,t). By previous Theorem, $W(x,x,t)=x$ so it suffices to show that $\{W(x_n,y_n,t_n)\}_{n=1}^{\infty}$ converges to x.Let $\varepsilon>0$.Since the sequences $\{x_n\}_{n=1}^{\infty}$ and $\{y_n\}_{n=1}^{\infty}$ both converge to x,there exists an $N\in\mathbb{N}$ such that $d(x_n,x)<\frac{\varepsilon}{2}$ and $d(y_n,x)<\frac{\varepsilon}{2}$ for all $n\geq N$.Now

$$\begin{aligned} d(x,W(x_n,y_n,t_n)) &\leq t_nd(x,x_n)+(1-t_n)d(x,y_n) \\ &\leq 1.d(x,x_n)+1.d(x,y_n) \\ &< \frac{\varepsilon}{2}+\frac{\varepsilon}{2} \text{ for all } n\geq N \\ &= \varepsilon. \end{aligned}$$

Thus $\{W(x_n,y_n,t_n)\}_{n=1}^{\infty}$ converges to x. □

Theorem 3.1.5. ([PhSu11], p. 3-4) *Let (X,d,W) be a WCM space, and let*

$(H) = d(W(x,y,t), W(x',y,t)) \leq t.d(x,x')$ *for all* $x, x', y \in X$ *and* $t \in [0,1]$.

$(H_1) \equiv d(W(x,y,t), W(x,y',t)) \leq (1-t).d(y,y')$ *for all* $x, y, y' \in X$ *and* $t \in [0,1]$.

$(S) \equiv d(W(x,y,t), W(u,v,t)) \leq t.d(x,u) + (1-t).d(y,v)$ *for all* $x, y, u, v \in X$ *and* $t \in [0,1]$.

$(C) \equiv W(x,y,t) = W(y,x,1-t)$ *for all* $x, y \in X$ *and* $t \in [0,1]$.

$(I) \equiv d(W(x,y,t), W(x,y,s)) \leq |t-s|\,.d(x,y)$.

Then (a) $(S) \Longrightarrow (H)$ and also $(S) \Longrightarrow ((H_1)$.

(b) If (C) holds, then $(H) \Leftrightarrow (H_1)$.

(c) (C) and $(H) \Longrightarrow (S)$; hence also (C) and $(H_1) \Longrightarrow (S)$.

(d) (C), (H) and $(I) \Longrightarrow W : X \times X \times I \longrightarrow X$ is jointly continuous.

Proof. (a) To show: $(S) \Longrightarrow (H)$, put $u = x'$ and $v = y$ in (S). Then

$$\begin{aligned} d(W(x,y,t), W(x',y,t)) &\leq t.d(x,x') + (1-t)d(y,y) \quad \text{(by } (S)\text{)} \\ &= t.d(x,x'). \end{aligned}$$

To show: $(S) \Longrightarrow (H_1)$, put $u = x$ and $v = y'$ in (S). Then

$$\begin{aligned} d(W(x,y,t), W(x,y',t)) &\leq td(x,x) + (1-t)d(y,y') \quad \text{(by } (S)\text{)} \\ &= (1-t)d(y,y'). \end{aligned}$$

(b) First (C) and $(H) \Rightarrow (H_1)$:

$$\begin{aligned} d(W(x,y,t), W(x,y',t)) &\leq d(W(y,x,1-t), W(y',x,1-t)) \text{ by (C)} \\ &\leq (1-t).d(y,y') \text{ (by } (H)\text{)}. \end{aligned}$$

Next (C) and $(H_1) \Rightarrow (H)$:

$$\begin{aligned} d(W(x,y,t), W(x',y,t)) &= d(W(y,x,1-t), W(y,x',1-t)) \text{ by (C)} \\ &\leq [1-(1-t)].d(x,x') \text{ (by } (H_1)\text{)}. \\ &= t.d(x,x') \end{aligned}$$

(c) To show: (C) and $(H) \Longrightarrow (S)$, let $x, y, u, v \in X$ and $t \in [0,1]$.Then

$$\begin{aligned} d(W(x,y,t), W(u,v,t)) &\leq d(W(x,y,t), W(u,y,t)) + d(W(u,y,t), W(u,v,t)) \\ &\leq d(W(x,y,t), W(u,y,t)) \\ &\quad + d(W(y,u,1-t), W(v,u,1-t)) \text{ (by } (C)\text{)}. \\ &\leq t.d(x,u) + [1-(1-t)]d(y,v) \text{ (by } (H)\text{)}. \\ &= td(x,u) + (1-t)d(y,v). \end{aligned}$$

Similarly, (C) and $(H_1) \Longrightarrow (S)$. In fact, if (C) holds, then $(H) \Leftrightarrow (H_1)$ by (b); hence, by above argument, (C) and $(H_1) \Longrightarrow (S)$.

(d) To show: (C), (H) and $(I) \Longrightarrow W : X \times X \times I \longrightarrow X$ is jointly continuous. Let $(x_0, y_0, t_0) \in X \times X \times I$, and let $\varepsilon > 0$. First: we define a metric d^* on $X \times X \times I$ as mapping $d^* : (X \times X \times I) \times (X \times X \times I) \longrightarrow \mathbb{R}$ by

$$\begin{aligned} d^*((x, y, t), (x', y', t')) &= d(x, x') + d(y, y') + \left|t - t'\right| \\ &\approx \max\{d(x, x'), d(y, y'), \left|t - t'\right|\} \end{aligned}$$

for all $x, x', y, y' \in X$ and $t, t' \in [0, 1]$. We want to prove that $\exists\ \delta > 0$ such that

$$d(W(x, y, t), W(x_{0,}y_0, t_0)) < \varepsilon \text{ if } d^*((x, y, t), (x_0, y_0, t_0)) < \delta.$$

We claim that for some constant $A, B, C > 0$,

$$d(W(x, y, t), W(x_{0,}y_0, t_0)) \leq Ad(x, x') + Bd(y, y') + C\left|t - t'\right|$$

Now,

$$\begin{aligned} & d(W(x, y, t), W(x_{0,}y_0, t_0)) \\ \leq\ & d(W(x, y, t), W(x_0, y, t)) + d(W(x_0, y, t), W(x_0, y_0, t)) \\ & + d(W(x_0, y_0, t), W(x_{0,}y_0, t_0)) \\ \leq\ & d(W(y, x, 1-t), W(y, x_0, 1-t)) + (1-t)d(y, y_0) \\ & + |t - t_0|\, d(x_0, y_0) \quad (\text{by } (C), (H) \text{ and } (I)). \\ \leq\ & [1 - (1-t)]d(x, x_0) + (1-t)d(y, y_0) + |t - t_0|\, d(x_0, y_0) \quad (\text{by } (H)). \\ =\ & td(x, x_0) + (1-t)d(y, y_0) + |t - t_0|\, d(x_0, y_0). \\ =\ & Ad(x, x_0) + Bd(y, y_0) + C\,|t - t_0| \ \ (A = t,\ B = (1-t), C = d(x_0, y_0)) \\ \leq\ & Kd(x, x_0) + Kd(y, y_0) + K|t - t_0| \quad (K = \max\{A, B, C\}) \\ =\ & K[d(x, x_0) + d(y, y_0) + |t - t_0|]. \\ =\ & Kd^*((x, y, t), (x_0, y_0, t_0)). \end{aligned} \tag{1}$$

Take $\delta = \frac{\varepsilon}{K}$. if $d^*((x, y, t), (x_0, y_0, t_0)) < \delta$, then by (1),

$$d(W(x, y, t), W(x_{0,}y_0, t_0)) < K.\delta = K.\frac{\varepsilon}{K} = \varepsilon.$$

Hence W is continuous at (x_0, y_0, t_0). $\square$

3.2 Uniform Convexity in WCM Spaces

In this section, we make a study of uniform convexity in WCM spaces as introduced by Shimizu and Takahashi in [ST96]. Also we present some properties of WCM spaces so that uniform convexity satisfy some of them. Further, we consider the relation between WCM spaces and Banach spaces for uniform convexity property. We shall see that in a uniformly WCM space (X, d, W), properties (C) and (I) always hold; further each of $(H), (H_1)$ implies that (S) holds.

Definition: A WCM space (X, d, W) is called a **uniformly** WCM **space** if for any $\varepsilon > 0$, $\exists\ \delta = \delta(\varepsilon) > 0$ such that for all $r > 0$ and $x, z_1, z_2 \in X$ with $d(x, z_1) \leq r, d(x, z_2) \leq r$,

$$d(z_1, z_2) \geq r.\varepsilon \Longrightarrow d(x, W(z_1, z_2, \frac{1}{2})) \leq (1-\delta)r. \tag{UC}$$

Remark. In the definition of uniformlyWCM space, clearly $\varepsilon \in (0, 2]$ and $\delta \in (0, 1]$.

Proof. Indeed, from the definition,

$$0 < r.\varepsilon \leq d(z_1, z_2) \leq d(z_1, x) + d(x, z_2) \leq r + r = 2r.$$

Since $r > 0$, $\delta \leq 2$, and so $\varepsilon \in (0, 2]$. Also from the definition,

$$0 \leq d(x, W(z_1, z_2, \frac{1}{2})) \leq (1-\delta)r.$$

Since $r > 0$, $1 - \delta \geq 0$, and so $\delta \leq 1$. $\square$

Theorem 3.2.1. **(**[PhSu11], p. 3) *Any uniformly WCM space (X, d, W) satisfies the property (C):*

$$W(x, y, t) = W(y, x, 1-t) \ \ \textit{for all } x, y \in X,\ t \in [0, 1]. \tag{C}$$

Proof. First we note that if $t = 0$ or $t = 1$ or $x = y$,then (C) holds trivially by using Theorem 3.1.3(a), as follows:

$$\begin{aligned} \text{If } t &= 0, \quad W(x, y, 0) = y,\ W(y, x, 1) = y; \\ \text{If } t &= 1, \quad W(x, y, 1) = x,\ W(y, x, 0) = x; \\ \text{If } x &= y, \quad W(x, x, t) = x,\ W(x, x, 1-t) = x. \end{aligned}$$

Now, suppose $0 < t < 1$ and $x \neq y$. From Theorem 3.1.3(b), we recall that:

$$\begin{aligned} d(x, W(x, y, t)) &= (1-t)d(x, y),\ d(y, W(x, y, t)) = td(x, y), && (1) \\ d(x, W(y, x, 1-t)) &= (1-t)d(x, y),\ d(y, W(y, x, 1-t)) = td(x, y). && (2) \end{aligned}$$

Suppose (C) does not hold. i.e., $W(x, y, t) \neq W(y, x, 1-t)$. Put

$$\begin{aligned} z_1 &= W(x, y, t),\ \ z_2 = W(y, x, 1-t); \\ r_1 &= (1-t)d(x, y) > 0,\ \ r_2 = td(x, y) > 0; \\ \varepsilon_1 &= \frac{d(z_1, z_2)}{r_1} > 0 \text{ and } \varepsilon_2 = \frac{d(z_1, z_2)}{r_2} > 0 \ \ (\text{since } z_1 \neq z_2). \end{aligned}$$

Then, by (1)-(2),

$$\begin{aligned} d(x,z_1) &= d(x,W(x,y,t)) = (1-t)d(x,y) = r_1 \le r_1, \\ d(x,z_2) &= d(x,W(y,x,1-t)) = (1-t)d(x,y) = r_1 \le r_1, \\ d(z_1,z_2) &= r_1\varepsilon_1 \ge r_1\varepsilon_1. \end{aligned}$$

Hence, by $(UC\ )$,$\exists\ \delta_1 = \delta(\varepsilon_1) > 0$ such that

$$d(x, W(z_1, z_2, \frac{1}{2})) \le (1-\delta_1)r_1. \tag{3}$$

Similarly,

$$\begin{aligned} d(y,z_1) &= d(y,W(x,y,t)) = td(x,y) = r_2 \le r_2, \\ d(y,z_2) &= d(y,W(y,x,1-t)) = td(y,x) = r_2 \le r_2, \\ d(z_1,z_2) &= r_2\varepsilon_2 \ge r_2\varepsilon_2. \end{aligned}$$

Hence, by $(UC\ )$,$\exists\ \delta_2 = \delta(\varepsilon_2) > 0$ such that

$$d(y, W(z_1, z_2, \frac{1}{2})) \le (1-\delta_2)r_2. \tag{4}$$

By using (3) and (4),

$$\begin{aligned} d(x,y) &\le d(x,W(z_1,z_2,\frac{1}{2})) + d(y,W(z_1,z_2,\frac{1}{2})) \\ &\le (1-\delta_1)r_1 + (1-\delta_2)r_2 \\ &< r_1 + r_2 = d(x,y). \end{aligned}$$

This is a contradiction. Thus (C) holds. □

Theorem 3.2.2. ([PhSu11], p. 4) *In a uniformly WCM space (X,d,W), the property $(H) \Rightarrow (S)$ and also $(H_1) \Rightarrow (S)$, where*

$$\begin{aligned} d(W(x,y,t),W(x',y,t)) &\le td(x,x') \text{ for all } x,y,y' \in X,\ t\in[0,1]; && (H) \\ d(W(x,y,t),W(x,y',t)) &\le (1-t)d(y,y') \text{ for all } x,y,y' \in X,\ t\in[0,1]; && (H_1) \\ d(W(x,y,t),W(u,v,t)) &\le td(x,u)+(1-t)d(y,v) \text{ for all } x,y,u,v \in X,\ t\in[0,1]. && (S) \end{aligned}$$

Proof. Let $x,y,u,v \in X$ and $t \in [0,1]$. Then, using property (H) and property (C) (which holds by Theorem 3.2.1),

$$\begin{aligned} & d(W(x,y,t),W(u,v,t)) \\ \le\ & d(W(x,y,t),W(u,y,t)) + d(W(u,y,t),W(u,v,t)) \\ \le\ & d(W(x,y,t),W(u,y,t)) + d(W(y,u,1-t),W(v,u,1-t)) \quad \text{(by } (C)) \\ \le\ & td(x,u) + (1-t)d(y,v) \quad \text{(by } (H)). \end{aligned}$$

Similarly, $(H_1) \Rightarrow (S)$.

Theorem 3.2.3. ([KaePa08], p. 8) *Any uniformly WCM space (X, d, W) satisfies the property (I):*

$$d(W(x,y,t), W(x,y,s)) \leq |t-s|\, d(x,y) \text{ for all } x, y \in X, s, t \in [0,1]. \qquad (I)$$

Proof. Let $x, y \in X$ and $s, t \in [0,1]$. If $t = 0$ or $s = 0$, then (I) holds by Theorem 3.1.3, as follows:

If $t = 0$,

$$d(W(x,y,0), W(x,y,s)) = d(y, W(x,y,s)) = sd(x,y) = |0-s|\, d(x,y);$$

if $s = 0$,

$$d(W(x,y,t), W(x,y,0)) = d(W(x,y,t), y) = td(x,y) = |t-0|\, d(x,y)$$

Suppose $t, s \in (0,1]$. We may assume that $t < s$. Put

$$\begin{aligned} u &= W(x,y,1-t), z = W(x,y,1-s), \\ \text{and let } v &= W(x,z,1-\frac{t}{s}). \end{aligned}$$

Then, using Theorem 3.1.3 (b) ,

$$\begin{aligned} d(x,v) &= d(x, W(x,z,1-\frac{t}{s})) \\ &= [1-(1-\frac{t}{s})]d(x,z) = \frac{t}{s}d(x,z) \\ &= \frac{t}{s}d(x, W(x,y,1-s)) \\ &= \frac{t}{s}[1-(1-s)]d(x,y) = td(x,y). \end{aligned}$$

So,

$$d(x,v) = \frac{t}{s}d(x,z) = td(x,y). \qquad (1)$$

Next, using again Theorem 3.1.3 (b),

$$\begin{aligned} d(y,v) &= d(y, W(x,z,1-\frac{t}{s})) \\ &\leq d(y,z) + d(z, W(x,z,1-\frac{t}{s})) \\ &= d(y, W(x,y,1-s)) + (1-\frac{t}{s})d(x,z) \\ &= (1-s)d(y,x) + (1-\frac{t}{s})d(x, W(x,y,1-s)) \\ &= (1-s)d(x,y) + (1-\frac{t}{s})[1-(1-s)]d(x,y) \\ &= (1-s)d(x,y) + (1-\frac{t}{s})sd(x,y) \\ &= [1-s+s-\frac{t}{s}.s]d(x,y) = (1-t)d(x,y). \end{aligned}$$

Also,

$$\begin{aligned}(1-\frac{t}{s})d(x,y)+\frac{t}{s}d(y,z) &= (1-\frac{t}{s})d(x,y)+\frac{t}{s}d(y,W(x,y,1-s))\\ &= (1-\frac{t}{s})d(x,y)+\frac{t}{s}(1-s)d(x,y)\\ &= (1-\frac{t}{s}+\frac{t}{s}-\frac{t}{s}.s)d(x,y)=(1-t)d(x,y).\end{aligned}$$

So,

$$d(y,v)\le(1-t)d(x,y)=(1-\frac{t}{s})d(x,y)+\frac{t}{s}d(y,z). \tag{2}$$

Next, using uniform W-convexity, we prove that $u=v$. Suppose $u\neq v$, and put

$$p=W(u,v,\frac{1}{2}),r=td(x,y) \text{ and } r'=(1-t)d(x,y).$$

First using $(UC\ )$, we show that (i) $d(x,p)<d(x,u)=r$,(ii) $d(y,p)<d(y,u)=r'$. Now, using Theorem 3.1.3(b).

$$d(x,u)=d(x,W(x,y,1-t))=[1-(1-t)]d(x,y)=td(x,y)=r\quad\text{(say)}.$$

Also, by (1),

$$d(x,v)=td(x,y)=r.$$

Since $u\neq v$, $d(u,v)\neq 0$. Taking $\varepsilon=\frac{d(u,v)}{r}>0$, we have

$$d(u,v)=r.\varepsilon\ge r.\varepsilon.$$

By $(UC\ )$, $\exists\ \delta=\delta(\varepsilon)>0$, $0<\delta<1$, such that

$$\begin{aligned} d(x,W(u,v,\frac{1}{2})) &\le r(1-\delta),\\ \text{or}\ \ d(x,p) &\le r(1-\delta)<r=d(x,u). \end{aligned}\tag{3}$$

Now, using Theorem 3.1.3(b).

$$d(y,u)=d(y,W(x,y,1-t))=(1-t)d(x,y)=r'\text{ (say)}.$$

Also, by (2),

$$d(y,v)\le(1-t)d(x,y)=r'$$

Taking $\varepsilon'=\frac{d(u,v)}{r'}>0$, $d(u,v)=r'.\varepsilon'\ge r'.\varepsilon'$. So by $(UC\ )$, $\exists\ \delta'=\delta'(\varepsilon')>0$, $0<\delta'<1$, such that

$$d(y,W(u,v,\frac{1}{2}))\le r'(1-\delta').$$

Or

$$d(y,p)\le r'(1-\delta')<r'=d(y,u) \tag{4}$$

Using (3) and (4)

$$\begin{aligned} d(x,y) &\leq d(x,p)+d(p,y) < r+r' \\ &= td(x,y)+(1-t)d(x,y) = d(x,y). \end{aligned}$$

This is a contradiction. Thus $u = v$.

Finally, since $t < s$,

$$\begin{aligned} d(u,z) &= d(v,z) = d(z,v) = d(z, W(x, z.1-\frac{t}{s})) \\ &= (1-\frac{t}{s})d(x,z) = (1-\frac{t}{s})d(x, W(x,y,1-s)) \\ &= (1-\frac{t}{s})[1-(1-s)]d(x,y) \\ &= \frac{s-t}{s}.sd(x,y) = (s-t)d(x,y) \\ &= |s-t|\, d(x,y) = |t-s|\, d(x,y). \end{aligned}$$

Or

$$d(W(x,y,1-t), W(x,y,1-s)) \leq |t-s|\, d(x,y).$$

Replacing $1-t$ by t and $1-s$ by s, we have

$$\begin{aligned} d(W(x,y,t), W(x,y,s)) &\leq |[(1-(1-t))-(1-(1-s))]|\, d(x,y) \\ &= |t-s|\, d(x,y). \ \square \end{aligned}$$

Theorem 3.2.4. **(**[PhSu11], p. 3**)** *Uniformly convex Banach spaces are uniformly WCM spaces.*

Proof. As in Section 3.1, $(X, \|.\|)$ is a WCM space by defining $W : X \times X \times I \longrightarrow X$ by

$$W(x,y,t) = tx + (1-t)y,\ x,y \in X, t \in [0,1].$$

Now $(X, \|.\|)$ is a uniformly convex means given $\varepsilon > 0$, there exists a $\delta > 0$ such that for all $x, y \in X$ with$\|x\| \leq 1, \|y\| \leq 1$,

$$\|x-y\| \geq \varepsilon \Rightarrow \left\|\frac{x+y}{2}\right\| \leq (1-\delta). \tag{1}$$

Let $z \in X$ .Change $x \to z-x, y \to z-y, 1 \to r$ and $\varepsilon \to r\varepsilon$.Thus $\|z-x\| \leq r, \|z-y\| \leq r$ and $\|(z-x)-(z-y)\| \geq r\varepsilon$. Then by (1)

$$\left\|\frac{(z-x)+(z-y)}{2}\right\| = \left\|z - \frac{x+y}{2}\right\| \leq (1-\delta)r. \tag{2}$$

Now

$$\begin{aligned} d(z, W(x,y,\frac{1}{2})) &= \left\|z - W(x,y,\frac{1}{2})\right\| = \left\|z - [\frac{1}{2}x + (1-\frac{1}{2})y)]\right\| \\ &= \left\|z - \frac{x+y}{2}\right\| \leq (1-\delta)r, \quad \text{by (2)}. \end{aligned}$$

Thus X is uniformlyWCM space. $\square$

3.3 Fixed Points and T-Invariant Approximation in WCM Spaces

In this section, we present some results of Beg-Shahzad-Iqbal [BSI92] on Fixed Points, Best Approximation and Invariant Approximation on a WCM spaces.

Definition: Let (X, d) be a metric space and $T : X \to X$ a mapping with $Fix(T) \neq \varnothing$. Then X is said to have the T-**invariant approximation property** if, for any $C \subseteq X$ with $T(C) \subseteq C$ and any $u \in Fix(T)$,

$$P_C(u) \cap Fix(T) \neq \varnothing.$$

i.e. $\exists$ a $z \in P_C(u)$ such that $T(z) = z$. In this case we also say that X has T-**invariant approximation.**

Definition: Let (X, d, W) be a WCM space. A subset K of X is said to be p-**starshaped** if there exists $p \in K$ such that $W(x, p, t) \in K$ for all $x \in K$ and $t \in [0, 1]$. Clearly p-starshaped subsets of X contain all convex subsets of X as a proper subclass.

Recall from Section 3.1 that (X, d, W) is said to satisfy **property** (H) if

$$d(W(x, p, t), W(y, p, t)) \leq td(x, y) \text{ for all } x, y, p \in X, t \in [0, 1]. \qquad (H)$$

Note. Property (H) is always satisfied in any normed space $X = (X, ||.||)$.

Proof. Let $x, y, p \in X$ and $t \in I$. In this case, $W(x, y, t) = tx + (1 - t)y$. Hence

$$\begin{aligned} d(W(x, p, t), W(y, p, t)) &= ||tx + (1 - t)p - ty - +(1 - t)p|| \\ &= t||x - y|| = td(x, y). \end{aligned}$$

For more properties, we refer to Guay, Singh and Whitfield [GSW82].

Definition: Let K be a nonempty subset of a convex metric space X. A mapping $T : K \to K$ is said to be a **Banach operator** if there exists a constant α such that $0 \leq \alpha < 1$ and for each $x \in K$

$$d(T^2x, Tx) \leq \alpha d(Tx, x).$$

Every contraction mapping is a Banach operator but not conversely. A Banach operator need not be continuous, nor need its fixed points be unique.

Theorem 3.3.1. *Let C be a closed subset of a metric space (X, d), and let $T : C \to C$ be a continuous Banach operator. Then T has a fixed point in C each of the following cases:*

(a) *(X, d) is complete.*

(b) *$\overline{T(C)}$ is compact.*

Proof. See Section 2.1.

Theorem 3.3.2. [BSI92, p. 98] *Let (X, d, W) be a WCM space with W continuous and satisfying property (H) and C be a closed and p-starshaped subset of X. If T is a nonexpansive self mapping on C and $\overline{T(C)}$ is compact, then T has a fixed point.*

Proof. Define a sequence of maps G_n,

$$G_n x = W(Tx, p, t_n),$$

where $\{t_n\} \subseteq \mathbb{R}$ is a fixed sequence with $0 \leq t_n < 1$ and $t_n \to 1$ (e.g., $t_n = \frac{n}{n+1}$). Each G_n maps C into itself because $T : C \to C$ and C is p-starshaped. Moreover each G_n is a continuous Banach operator:

$$\begin{aligned} d(G_n x, G_n^2 x) &= d(W(Tx, p, t_n), W(T(G_n x), p, t_n)) \\ &\leq t_n d(Tx, T(G_n x)) \quad \text{(by } (H)) \\ &\leq t_n d(x, G_n x). \quad \text{(since } T \text{ is nonexpansive)} \end{aligned}$$

Since $\overline{T(C)}$ is compact, $\overline{G_n(C)}$ is compact too, and previous theorem further implies: for each G_n there exists a fixed point x_n such that $x_n = G_n x_n = W(Tx_n, p, t_n)$.As $\overline{G_n(C)}$ is compact and $G_n(C) \subseteq \overline{G_n(C)}$, $\{W(Tx_n, p, t_n)\}$ has a subsequence $\{W(Tx_{n_i}, p, t_n)\} = G_{n_i} x_{n_i}$ converging, e.g., to y.

$$\lim_{i \to \infty} W(Tx_{n_i}, p, t_n) = \lim_{i \to \infty} G_{n_i} x_{n_i} = y. \tag{1}$$

By continuity of T and W,

$$\lim_{i \to \infty} W(Tx_{n_i}, p, t_n) = W(T(\lim_{i \to \infty} x_{n_i}), p, \lim_{i \to \infty} t_n) = W(Ty, p, 1) = Ty. \tag{2}$$

From (1) & (2), $Ty = y$. $\square$

Remark. As immediate corollary to Theorem 3.3.2, we have Theorem 4 of [Hab89].

Best Approximation

Let C be a closed convex subset of a WCM space (X, d, W). The a function $S : C \to C$ into itself is said to be **affine** if

$$S(W(x, y, t)) = W(Sx, Sy, t)$$

whenever $t \in [0, 1] \cap \mathbb{Q}$ and x, y in C, where $\mathbb{Q}$ denotes, the set of rational numbers.

Theorem 3.3.3. [Jung76] *Let (X, d) be a compact metric space and $T, S : X \to X$ be two commuting mappings such that $T(X) \subseteq S(X)$, S is continuous and $d(Tx, Ty) < d(Sx, Sy)$ whenever $Sx \neq Sy$. Then $Fix(T) \cap Fix(S)$ is a singleton.*

Theorem 3.3.4. [BSI92, p. 99] *Let (X, d, W) be a compact WCM space with W continuous and satisfying condition (H). Let $T, S : X \to X$ be operators, C be a* *subset of X such that $T : \partial C \to C$ and $x^* \in Fix(T) \cap Fix(S)$. Further T and S satisfy*

$$d(Tx, Ty) \leq d(Sx, Sy)$$

for all x and y in $P_C(x^*) \cup \{x^*\}$, and let S be continuous and affine on $P_C(x^*)$ and $STx = TSx$ for all x in $P_C(x^*)$. If $P_C(x^*)$ is nonempty, compact and q-starshaped with respect to a point $q \in Fix(S)$ and if $S(P_C(x^*)) = P_C(x^*)$, then

$$P_C(x^*) \cap Fix(T) \cap Fix(S) \neq \varnothing.$$

Proof. If $y \in P_C(x^*)$, then

$$\begin{aligned} d(Ty, x^*) &= d(Ty, Tx^*) \text{ (since } x^* \in Fix(T)) \\ &\leq d(Ty, T(W(y, x^*, t))) + d(T(W(y, x^*, t)), Tx^*) \\ &\leq d(Sy, S(W(y, x^*, t))) + d(S(W(y, x^*, t)), Sx^*) \\ &= d(Sy, Sx^*) \text{ (since } X \text{ is WCM)} \\ &= d(Sy, x^*) \text{ (since } x^* \in Fix(S)) \\ &= d(x^*, C) \text{ (since } S(P_C(x^*)) = P_C(x^*),\ Sy \in P_C(x^*)). \end{aligned}$$

Thus we obtain that $Ty \in P_C(x^*)$ (since $Ty \in C$). Thus T maps $P_C(x^*)$ into itself.

Let $\{k_n\}$ be a sequence, $k_n = 1 - \frac{1}{n}$ and with $k_n \to 1$. Define $T : P_C(x^*) \to P_C(x^*)$ as

$$G_n(x) = W(Tx, q, k_n) \text{ for all } x \in P_C(x^*).$$

Since S is affine and commutes with T on $P_C(x^*)$, we have

$$\begin{aligned} G_nSx &= W(TSx, Sq, k_n) \text{ (since } q \in Fix(S)) \\ &= W(STx, Sq, k_n) \\ &= SW(Tx, q, k_n) = SG_nx. \end{aligned}$$

Thus S commutes with T for each n for all x in $P_C(x^*)$ and $G_n(P_C(x^*)) \subseteq P_C(x^*) = S(P_C(x^*))$.Furthermore, by (H),

$$\begin{aligned} d(G_nx, G_ny) &= d(W(Tx, q, k_n), W(Ty, q, k_n)) \\ &\leq k_nd(Tx, Ty) \leq k_nd(Sx, Sy) < d(Sx, Sy), \end{aligned}$$

whenever $Sx \neq Sy$. By Theorem 3.3.3, we have $Fix(G_n) \cap Fix(S) = \{x_n\}$ for each n. Since $P_C(x^*)$ is compact, $\{x_n\}$ has a subsequence $x_{n_i} \to z$, (say). Now $x_{n_i} = G_{n_i}x_{n_i} = W(Tx_{n_i}, q, k_{n_i})$.Since $k_{n_i} \to 1$, then

$$\lim_{i\to\infty} x_{n_i} = \lim_{i\to\infty} W(Tx_{n_i}, q, k_{n_i}) = W(Tz, q, 1) = Tz$$

Hence $Tz = z$. Thus $z \in P_C(x^*) \cap Fix(T)$. The continuity of S further implies that

$$Sz = S[\lim_{i\to\infty} x_{n_i}] = \lim_{i\to\infty} x_{n_i} = z,$$

that is, $z \in Fix(S)$. $\square$

Remark. Theorem 3.3.4 generalized serval known results including among these are Brosowski [Bros69] Sahab and Khan [SK88] and Singh [Singh79b].

Invariance of Best Approximation

In [Mein63] Meinardus introduced the notion of invariant approximation. The aim of this section is to prove some results regarding invariant approximation in WCM spaces.

Definition: A metric space X is called γ**-chainable** if for every $x, y \in X$, there exists an γ-chain that is a finite set of points $x = x_0, x_1, x_2, ..., x_n = y$ (n may depend on x and y) such that $d(x_{i-1}, x_i) \leq \gamma$, $(i = 1, 2, ..., n)$.

Theorem 3.3.5. [BSI92, p. 100] *Let (X, d, W) be a γ-chainable WCM space with* W continuous and *satisfying property (H) and $T : X \to X$ be a mapping. Let C be a subset of X such that $T(C) \subseteq C$ and $x_0 \in Fix(T)$. If $P_C(x_0)$ is nonempty, compact, and p-starshaped and T is:*

(i) continuous on $P_C(x_0)$ and

(ii) $d(x, y) \leq d(x_0, C)$ implies $d(Tx, Ty) \leq d(x, y)$ for all $x, y \in P_C(x_0) \cup \{x_0\}$.

Then $Fix(T) \cap P_C(x_0) \neq \varnothing$; i.e. *$C$ contains a T-invariant point, which is a best approximation to x_0 in C.*

Proof. Similar to the proof of Theorem 3.3.4, only need to notice that the corresponding G_n are uniformly locally contractive and have fixed point x_n from Theorem 5.2 of [Ed61]. $\square$

Remark. With the notion of convex metric space, our Theorem 3.3.5 generalizes Theorem 1 of Singh [Singh79b].

If an operator $T : X \to X$ leaves a subset Y of X invariant, then a restriction of T to Y will be denoted by the symbol T/Y.

Theorem 3.3.6. [BSI92, p. 100] *Let (X, d, W) be a WCM space with W continuous and satisfying property (H) and T be a nonexpansive mapping on X. Let C be a T-invariant subset of X and T/C be compact and x be a $T-$invariant point. If the set of best $C-$approximants to x is nonempty, convex and compact, then it contains a T-invariant point.*

Proof. Let $K = P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$. Then the set K is T-invariant. To see this, let $z \in T(K)$, then $\exists$ an $s^* \in K$ such that $z = T(s^*)$. Now

$$d(x, z) = d(x, T(s^*)) = d(T(x), T(s^*)) \leq d(x, s^*) = d(x, C).$$

Since C be a T-invariant subset of X , $z \in C$. Thus

$$d(x, z) = d(x, C),$$

hence $z \in K$. Also K is closed and convex. Since K is bounded subset of C and T/C is compact, the $\overline{T(K)}$ is compact. Theorem 3.3.2 implies that T has a fixed point in K. $\square$

3.4 Simultaneous Approximation in WCM Spaces

In this section we discuss the problem of best simultaneous approximation (b.s.a). Motivated by this problem if sets of elements A and B is given in metric space (X, d), one might like to approximate all the elements of B simultaneously by a single element of A. This type of problem arises when a function being approximated is not known precisely but is known to belong to a set. Several mathematicians have studied this problem of simultaneous approximation in normed linear spaces. We will study this problem in WCM spaces as given by Narang in [Nar99].

Let $K \subseteq (X, d)$. Recall that: An element $a_0 \in K$ is called a **best approximant** to x in K if

$$d(x, a_0) = d(x, K).$$

Definition: Let $K \subseteq (X, d)$. Given any bounded subset F of X, define

$$\delta(F, K) = \sup_{y \in F} d(y, K) = \sup_{y \in F} \inf_{x \in K} d(y, x).$$

An element $k^* \in K$ is said to be a **best simultaneous approximation** (b.s.a) to F if

$$\sup_{y \in F} d(y, k^*) = \delta(F, K).$$

C. B. Dunham, J. B. Diaz and H. W. McLaughlin have considered the problem of best simultaneous approximation in the following case: $X = C\ [a, b]$, K a non empty subset of X and $F = \{p_1, p_2\}$.Goel, Holland, Nasim, and Sahney studied the problem when X is a normed linear space, K any subset of X and $F = \{x_1, x_2\} \subseteq X$. Using the same procedure as in [GHNS74] it is possible to study the problem when $F = \{x_1, x_2, ..., x_n\}$. Holland Sahney and Tzimbalario studied the problem when F is a compact subset of a normed linear space.

Now we recall a few definitions.

Definition: A bounded subset F of a metric space (X, d) is said to be **remotal** with respect to a subset K of X if for each $k \in K$ there exists a point $p_0 \in F$ farthest from k,i.e.

$$d(k, p_0) \geq d(k, p) \text{ for all } p \in F.$$

Definition: A WCM spaces (X, d) is said to be **strictly W-convex** [Nar81] if for every $x, y \in X$ and $r > 0$,

$$d(u, x) \leq r, d(u, y) \leq r \text{ imply } d(u, W(x, y, t)) < r \text{ unless } x = y$$

where u is arbitrary but fixed point of X.

We show that the problem of b.s.a is equivalent to the problem of minimizing certain functional. For this we prove a lemma

Lemma 3.4.1. ([Nar99], p. 62) *Let K be any subset of a metric space (X, d) and F a bounded subset of X. Then the functional $\varphi : K \to \mathbb{R}$ defined by*

$$\varphi(k) = \sup_{p \in F} d(p, k), \;\; k \in K,$$

is continuous.

Proof. Let $\varepsilon > 0$ be given. For any $p \in F$ and $k, k' \in K$ we have

$$d(p, k) \leq d(p, k') + d(k, k')$$

and so

$$\sup_{p \in F} d(p, k) \leq \sup_{p \in F} d(p, k') + d(k, k')$$

i.e.

$$\varphi(k) - \varphi(k') \leq d(k, k').$$

Interchanging k and k', we get

$$\varphi(k') - \varphi(k) \leq d(k, k').$$

and so

$$\left|\varphi(k) - \varphi(k')\right| \leq d(k, k').$$

Therefore if $d(k, k') < \varepsilon$ then $|\varphi(k) - \varphi(k')| < \varepsilon$ and so φ is continuous. $\square$

Note. In normed vector spaces this lemma was proved in [HST76].

If there exists a $k^* \in K$ such that $\varphi(k^*) = \inf\limits_{k \in K} \varphi(k)$ then $k^* \in K$ is a b.s.a to F. So the problem of b.s.a reduces to the problem of minimizing the functional on K and so we have

Theorem 3.4.2. ([Nar99], p. 62) *Let F be a bounded subset of a metric space (X, d) and K a subset of X such that the continuous functional $\varphi : K \to \mathbb{R}$ defined by $\varphi(k) = \sup_{p \in F} d(p, k)$ attains its infimum at some point of K then there always exists a b.s.a in K to F.*

Since for compact sets K and for sets K which are approximatively compact with respect to F (i.e. any sequence (k_n) in K satisfying $\sup_{p \in F} d(k_n, p) \to \delta(F, K)$ is compact in K), we can find $k^* \in K$ such that $\varphi(k^*) = \inf_{k \in K} \varphi(k)$, we have

Corollary 3.4.3. ([Nar99], p. 63) *Let K be a compact subset of a metric space (X, d) and F be any bounded subset of X Then there exists a b.s.a in K to F.*

Corollary 3.4.4. [Go84] *If F is a bounded subset of a metric space (X, d) and K is approximatively compact with respect to F then there exists a b.s.a in K to F*

The following result which generalizes Lemma 3 of [HST76] deals with the convexity of the set of best simultaneous approximants

Lemma 3.4.5. ([Nar99], p. 63) *Let K be a W-convex subset of a WCM space (X, d) and F a bounded subset of X. If $u, v \in K$ are b.s.a to F then $W(u, v, t)$ is also a b.s.a in K to F for every $t \in I$.*

Proof. Since $u, v \in K$ are b.s.a to F,

$$\sup_{p\in F} d(p, u) = \delta(F, K) = \sup_{p\in F} d(p, v).$$

For any $p \in F$, consider $d(p, W(u, v, t)) \leq td(p, u) + (1 - t)d(p, v)$.This implies

$$\begin{aligned} \sup_{p\in F} d(p, W(u, v, t)) &\leq t \sup_{p\in F} d(p, u) + (1 - t) \sup_{p\in F} d(p, v) \\ &= t\ \delta(F, K) + (1 - t)\delta(F, K) \\ &= \delta(F, K) \leq \sup_{p\in F} d(p, W(u, v, t)) \end{aligned}$$

as $W(u, v, t) \in K$ by the convexity of K. Therefore,

$$\sup_{p\in F} d(p, W(u, v, t)) = \delta(F, K),$$

proving thereby that $W(u, v, t)$ is a b.s.a in K to F for every $t \in I$. □

The following result deals with the uniqueness of b.s.a.

Theorem 3.4.6. ([Nar99], p. 63) *Let K be a W-convex subset of a strictly WCM space (X, d) and F be a subset of X which is remotal w.r.t. K. Then there exists at most one b.s.a in K to F.*

Proof. Suppose $u, v, (u \neq v)$ are two b.s.a in K to the set F, i.e.

$$\sup_{p\in F} d(p, u) = \delta(F, K) = \sup_{p\in F} d(p, v).$$

By Lemma 3.4.5, $W(u, v, t) \in K$ is also a b.s.a to F, i.e.

$$\sup_{p\in F} d(p, W(u, v, t)) = \delta(F, K)$$

for every $t \in I$. Let $t \in I$ be arbitrary. Keep it fixed. Since F is remotal w.r.t. K, there exists an element $p^* \in F$ such that

$$d(p^*, W(u, v, t)) = \sup_{p\in F} d(p, W(u, v, t)) = \delta(F, K). \tag{1}$$

Now $d(p^*, u) \leq \delta(F, K)$ and $d(p^*, v) \leq \delta(F, K)$ and since the space is strictly W-convex, we have

$$d(p^*, W(u, v, t)) < \delta(F, K) \text{ unless } u = v.$$

This contradicts (1) and hence the uniqueness. □

Combining Theorems 3.4.2 and 3.4.6 we have

Theorem 3.4.7. ([Nar99], p. 63) *Let K be a W-convex subset of a strictly WCM space (X, d) and F a subset of X which is remotal w.r.t. K If the continuous functional $\varphi : K \to \mathbb{R}$ defined by $\varphi(k) = \sup_{p \in F} d(p, k)$ attains its infimum on K then there exists a unique b.s.a in K to F.*

Since the functional φ attains its infimum when K is compact subset of a metric space (see [Nar83]) or K is approximatively compact w.r.t. a bounded subset F of a metric space (see [Go84]), we have

Corollary 3.4.8. ([Nar99], p. 64) *Let K be compact W-convex subset of a strictly WCM space(X, d) and F a subset of X which is remotal w.r.t K then there exists a unique b.s.a in K to F.*

Corollary 3.4.9. ([Nar99], p. 64) *Let F be a bounded subset of a strictly WCM space (X, d), K a W-convex subset of X which is appoximatively compact w.r.t. F and F is remotal w.r.t. K then there exists a unique b.s.a in K to F.*

Remarks. (1) Uniqueness of elements of b.s.a is also guaranteed if the functional φ defined in Lemma 3.4.1 attains its infimum at exactly one $k^* \in K$.

(2) When F is a singleton say $\{p\}$ then $\delta(F, K) = \inf_{k \in K} d(p, k) = d(p, K)$.So the problem of b.s.a reduces to the problem of best approximation and consequently, results proved in this section extends known results on best approximation.

3.5 Best Approximation and Fixed Points in WM-Starshaped Metric Spaces

In this section, we present some results of Al-Thagafi [AlThag95]. He considered more general notion of convexity, called WM-starshaped metric spaces by considering the map $W : X \times M \times I \to X$ instead of $W : X \times X \times I \to X$ where $M \subseteq (X, d)$. Further, he obtained some results on the existence of fixed points of best approximation.

Definitions: Let X be a metric space, $T : X \to X$, $D \subseteq X$, and $p \in X$. Then

(1) T is γ-**nonexpansive** on D if $d(Tx, Ty) \leq d(x, y)$ for every $x, y \in D$ with $d(x, y) \leq \gamma$.

(2) T is γ-**contraction** on D if $d(Tx, Ty) \leq \lambda d(x, y)$ for some $\lambda \in [0, 1)$ and for every $x, y \in D$ with $d(x, y) \leq \gamma$.

Recall from Section 3.3 that: D is a γ**-chainable** metric space if for any pair $x, y \in D$, there exists a finite chain of points $x_0, x_1, ..., x_n$ in D with $x_0 = x$ and $x_n = y$ such that $d(x_{i-1}, x_i) \leq \gamma$, for $i = 1, 2, ..., n$.

Remarks. (1) Brosowski [Bros69] proved that if X is a normed vector space, $T : X \to X$ is nonexpansive with a fixed point p, $K \subseteq X$ with $T(K) \subseteq K$, then T has a fixed point in $P_K(p)$ provided that $P_K(p)$ is nonempty, compact and convex.

(2) Singh ([Singh79a], Theorem 1) relaxed the convexity of $P_K(p)$ be starshapedness.

(3) However, Hicks and Humphries ([HH82], p. 221) showed that the conclusion of Singh's result still holds whenever $T(K) \subseteq K$ is replaced by $T(\partial K) \subseteq K$.

(4) Subrahmanyam ([Su77], Theorem 3) proved that if X is a normed vector space, $T : X \to X$ is $d(p, K)$-nonexpansive with a fixed point p, $K \subseteq X$ with $T(K) \subseteq K$, then T has a fixed point in $P_K(p)$ provided that K is a finite-dimensional subspace of X.

(5) However, Singh ([Singh79b], Theorem 1) showed that the conclusion of Subrahmanyam's result still holds whenever the finite-dimensionality of K is replaced by the following conditions:

(i) $P_K(p)$ is nonempty, compact and starshaped, and

(ii) T is continuous on $P_K(p)$.

Theorem 3.5.1. [Ed61] *Let D be a complete and γ-chainable metric space. If $T : D \to D$ is a γ-contraction, then T has a unique fixed point in D.*

Definition: Let X be a metric space, $M \subseteq X$ and $I = [0, 1]$.

(a) X is WM**-starshaped** if there exists a mapping $W : X \times M \times I \to X$, satisfying

$$d(z, W(x, q, t)) \leq td(z, x) + (1 - t)d(z, q)$$

for every $z, x \in X$, all $q \in M$ and all $t \in I$.

(b) X is **strong** WM**-starshaped** if it is WM-starshaped and W satisfies (H):

$$d(W(x, q, t), W(y, q, t)) \leq td(x, y), \tag{H}$$

for every $x, y \in X$, all $q \in M$ and all $t \in I$.

(c) (i) X is **strong convex** if it is strong X-starshaped.

(ii) X is **convex** if it is X-starshaped.

(iii) X is **starshaped** if it is $\{q\}$-starshaped for some $q \in X$.

Convex and starshaped metric spaces were introduced by Takahashi [Tak70]. Each normed vector space X is a strong convex metric space with W defined by

$$W(x, q, t) = tx + (1 - t)q \text{ for every } x, q \in X \text{ and } t \in I.$$

Definition: Let X be a WM-starshaped metric space. A subset D of X is **q-starshaped** if there exists $q \in D \cap M$ with $W(D \times \{q\} \times I) \subseteq D$. A q-starshaped subset of a convex metric space is called **starshaped**.

Theorem 3.5.2. ([AlThag95], p. 614) *Let X be a strong WM-starshaped metric space and $D \subseteq X$. If D is compact, γ-chainable and q-starshaped, and $T : D \to D$ is γ-nonexpansive, then T has a fixed point in D.*

Proof. For each positive integer n, let $t_n = \frac{n}{n+1}$ and $G_n x = W(Tx, q, t_n)$ for all $x \in D$. By the strong WM-starshapedness of X and the γ-nonexpansiveness of T on D, each G_n satisfies

$$\begin{aligned} d(G_n x, G_n y) &= d(W(Tx, q, t_n), W(Ty, q, t_n)) \\ &\leq t_n d(Tx, Ty) \leq t_n d(x, y) \end{aligned}$$

for every $x, y \in D$ with $d(x, y) \leq \gamma$. Clearly, $G_n : D \to D$ (since D is q-starshaped, $G_n x = W(Tx, q, t_n) \in D$ for all $x \in D$). So each G_n is a γ- contraction selfmap of D. Since D is γ-chainable, Theorem 3.5.1 shows that each G_n has a unique fixed point $x_n \in D$. By the compactness of D, there exists a subsequence $\{x_{n_i}\}$ of $\{x_n\}$ with $\lim_{i\to\infty} x_{n_i} = x_0 \in D$. Since

$$\begin{aligned} d(Tx_{n_i}, x_{n_i}) &= d(Tx_{n_i}, G_n x_{n_i}) = d(Tx_{n_i}, W(Tx_{n_i}, q, t_{n_i})) \\ &\leq (1 - t_{n_i}) d(Tx_{n_i}, q) \end{aligned}$$

for all i and $t_{n_i} \to 1$

$$\lim_{i\to\infty} d(Tx_{n_i}, x_{n_i}) \leq \lim_{i\to\infty} (1 - t_{n_i}) d(Tx_{n_i}, q) = 0$$

So $\lim_{i\to\infty} d(Tx_{n_i}, x_{n_i}) = 0$. Now, the γ-nonexpansiveness of T on D implies its continuity, so

$$\begin{aligned} \lim_{i\to\infty} d(Tx_{n_i}, x_{n_i}) &= 0 \Longrightarrow d(\lim_{i\to\infty} Tx_{n_i}, \lim_{i\to\infty} x_{n_i}) = 0 \\ &\Longrightarrow d(T(\lim_{i\to\infty} x_{n_i}), \lim_{i\to\infty} x_{n_i}) = 0 \Longrightarrow d(Tx_0, x_0) = 0. \end{aligned}$$

Hence x_0 is a fixed point of T. $\square$

The following is a result of Dotson ([Do73], Theorem 1).

Corollary 3.5.3. ([AlThag95], p. 614) *Let X be a normed vector space and $D \subseteq X$. If D is compact and starshaped and if $T : D \to D$ is nonexpansive, then T has a fixed point in D.*

Proof. Suppose D is p-starshaped for some $p \in D$. We will use Theorem 3.5.2 in the proof of this corollary. So we want to show that:

(a) X is strong $W\{p\}$-starshaped, (b) D is γ-chainable, (c) T is γ-nonexpansive.

(a) We consider $M = \{p\}$. Define a mapping $W : X \times \{p\} \times I \to X$ by

$$W(x, p, t) = tx + (1-t)p \text{ where } x \in X \text{ and } t \in I = [0, 1].$$

Clearly $W(x, p, t) \in X$ (since X is a normed vector space). Let $u, x \in X$ and $t \in [0, 1]$. Then,

$$\begin{aligned} d(u, W(x, p, t)) &= \|u - W(x, p, t)\| \\ &= \|[tu + (1-t)u] + [tx + (1-t)p]\| \\ &= \|t(u - x) + (1-t)(u-p)\| \\ &\leq t\|u - x\| + (1-t)\|u - p\| \\ &= td(u, x) + (1-t)d(u, p). \end{aligned}$$

Therefore, X is $W\{p\}$-starshaped. Let $x, y \in X$, $p \in M$ and $t \in [0, 1]$. Next,

$$\begin{aligned} d(W(x, p, t), W(y, p, t)) &= \|W(x, p, t) - W(y, p, t)\| \\ &= \|[tx + (1-t)p] - [ty + (1-t)p]\| \\ &\leq t\|x - y\| + (1-t)\|p - p\| \\ &= t\|x - y\| + 0 \\ &= td(x, y). \end{aligned}$$

Hence, X is strong $W\{p\}$-starshaped.

(b) D is γ-chainable: Let $\gamma > 0$, and $x \neq y \in D$. We need to find a finite chain of points $\{z_0, z_1, ..., z_k\}$ in D with $z_0 = x$ and $z_k = y$ such that $||z_{i-1} - z_i|| \leq \gamma$, for $i = 1, 2, ..., k$.

Suppose D is q-starshaped for some $q \in D$. Then $[x, q], [q, y] \subseteq D$. Now

$$\begin{aligned} [x, q] &= \{(1-t)x + tq : 0 \leq t \leq 1\} \\ &= \{x + t(q - x) : 0 \leq t \leq 1\}. \end{aligned} \tag{1}$$

Suppose $x \neq q$, and let $\gamma > 0$ and $c = ||q - x|| > 0$. Choose $N \geq \max\{2, \frac{2\gamma}{c}\}$, so that $\frac{2\gamma}{cN} \leq 1$, $\frac{2}{N} \leq 1$. Then, for each $i = 1, 2, ..., N$,

$$\frac{2\gamma i}{cN^2} = \frac{2\gamma}{cN} . \frac{i}{N} \leq 1.1 = 1,$$

and so, by (1),

$$x_i = x + \frac{2\gamma i}{cN^2}(q - x) \in [x, q].$$

Further,

$$\begin{aligned}
||x_{i-1} - x_i|| &= ||[x + \frac{2\gamma(i-1)}{cN^2}(q-x)] - [x + \frac{2\gamma i}{cN^2}(q-x)]|| \\
&= ||\frac{2\gamma i}{cN^2}(q-x) - \frac{2\gamma}{cN^2}(q-x) - \frac{2\gamma i}{cN^2}(q-x)]|| \\
&= ||-\frac{2\gamma}{cN^2}(q-x)|| = \frac{2\gamma}{cN^2}||q-x|| = \frac{2\gamma}{cN^2}.c \\
&= \frac{2}{N^2}.\gamma = \frac{2}{N}.\frac{1}{N}.\gamma \leq \gamma.
\end{aligned}$$

Hence $\{x_0, x_1, ..., x_N\}$ is a finite γ-chain in D for the pair (x, q) with $x_0 = x$ and $x_N = q$.

Similarly, we obtain a finite γ-chain $\{y_0, y_1, ..., y_M\}$ in D for the pair (q, y) with $y_0 = q$ and $y_M = y$. Thus

$$\{x_0, x_1, ..., x_N, y_1, ..., y_M\} = \{z_0, z_1, ..., z_k\}$$

is a finite chain in D for the pair (x, y). with $z_0 = x_0 = x$, $z_k = y_M = y$ and $||z_{i-1} - z_i|| \leq \gamma$, for $i = 1, 2, ..., k$.

(c) T is γ-nonexpansive: Let $\gamma > 0$, $x, y \in D$ with $d(x, y) \leq \gamma$. Now,

$$d(Tx, Ty) \leq d(x, y), \text{ (since } T \text{ is nonexpansive)}.$$

So, from (a), (b), (c), we can apply Theorem 3.5.2. Hence, T has a fixed point in D. □

Theorem 3.5.4. **(**[AlThag95], p. 615**)** *Let X be a strong WM-*starshaped *metric space and $D \subseteq X$. If D is compact and q-starshaped and if $T : D \to D$ is nonexpansive, then T has a fixed point in D.*

The proof of Theorem 3.5.4 is similar to the one given for Theorem 3.5.2; however, we use Banach's contraction principle instead of Theorem 3.5.1.

Note. The Corollary 3.5.3 follows also from Theorem 3.5.4.

Best Approximation in WM-Starshaped Metric Spaces

Lemma 3.5.5. **(**[AlThag95], p. 615**)** *Let X be a strong WM-*starshaped *metric space, $K \subseteq X$ and $p \in X$. If $P_K(P)$ is q-starshaped, then $P_K(p)$ is $d(p, K)$-chainable.*

Proof. For $x, y \in P_K(p)$. Let

$$x_0 = x,\ x_1 = W(x, q, \frac{1}{2}),\ x_2 = W(y, q, \frac{1}{2}),\ x_3 = y.$$

Since $P_K(p)$ is q-starshaped, then x_0, x_1, x_2, x_3 belongs to $P_K(P)$ and $q \in P_K(p)$. Now, the strong WM-starshapedness of X implies that

$$\begin{aligned}
d(x_0,x_1) &= d(x,W(x,q,\frac{1}{2})) \leq \frac{1}{2}d(x,q) \leq \frac{1}{2}[d(x,p)+d(p,q)] \\
&= \frac{1}{2}[d(p,K)+d(p,K)] = d(p,K), \\
d(x_1,x_2) &= d(W(x,q,\frac{1}{2}),W(y,q,\frac{1}{2})) \leq \frac{1}{2}d(x,y) \\
&\leq \frac{1}{2}[d(x,p)+d(y,p)] = \frac{1}{2}[d(p,K)+d(p,K)] = d(p,K), \\
d(x_2,x_3) &= d(W(y,q,\frac{1}{2}),y) \leq \frac{1}{2}d(y,q) \\
&\leq \frac{1}{2}[d(y,p)+d(q,p)] = \frac{1}{2}[d(p,K)+d(p,K)] = d(p,K).
\end{aligned}$$

Therefore $P_K(p)$ is $d(p,K)$-chainable. □

Lemma 3.5.6. ([AlThag95], p. 615) *Let X be a WM-*starshaped *metric space, $K \subseteq X$ and $p \in X$. Then $P_K(p) \subseteq \partial K \cap K$.*

Proof. Let $y \in P_K(p)$ and let $t_n = \frac{n}{n+1}$ for each positive integer n.Then

$$d(p,W(y,p,t_n)) \leq t_n d(p,y) < 1.d(p,K) = d(p,K) \text{ (since } t_n < 1).$$

which implies that $W(y,p,t_n) \notin K$ for every n (since $d(p,K) = \inf_{z\in K} d(p,z)$). Since

$$d(y,W(y,p,t_n)) \leq (1-t_n)d(y,p) = (1-t_n)d(p,K)$$

for all n, then

$$\begin{aligned}
\lim_{n\to\infty} d(y,W(y,p,t_n)) &\leq \lim_{n\to\infty}(1-t_n)d(p,K) \\
&= \lim_{n\to\infty}(1-(\frac{n}{n+1}))d(p,K) \\
&= 0.
\end{aligned}$$

So $\lim_{n\to\infty} d(y,W(y,p,t_n)) = 0$. Then $\lim_{n\to\infty} W(y,p,t_n) = y$. (i.e., for each $\varepsilon > 0$, $\exists$ an integer N such that $d(W(y,p,t_n),y) < \varepsilon$ for all $n \geq N$). Thus each neighborhood of y contains at least one $W(y,p,t_n)$ ($W(y,p,t_n) \in X \setminus K$). Hence $y \in \partial K$. □

Theorem 3.5.7. ([AlThag95], p. 615) *Let X be a strong WM-*starshaped *metric space, $T : X \to X$, $K \subseteq X$, and $p \in X$ a fixed point of T. If $P_K(p)$ is compact and q-starshaped, $T(K) \subseteq K$, as T is $d(p,K)$-nonexpansive on $P_K(P) \cup \{p\}$, then T has a fixed point in $P_K(p)$.*

Proof. Let $y \in P_K(p)$. Then $Ty \in K$ (since $P_K(p) \subseteq K$ and $T(K) \subseteq K$). and, by the $d(p,K)$-nonexpansiveness of T on

$P_K(p) \cup \{p\}$,

$$d(Ty,p) = d(Ty,Tp) \leq d(y,p) \text{ with } d(y,p) \leq d(p,K).$$

So $d(Ty, p) \leq d(p, K)$. But it is impossible that $d(Ty, p) < d(p, K)$ (since $Ty \in K$). Then $d(Ty, p) = d(p, K)$. Thus $Ty \in P_K(p)$ and so $T : P_K(p) \to P_K(p)$. Now, Theorem 3.5.2, with $D = P_K(p)$, and Lemma 3.5.5 show that T has a fixed point in $P_K(p)$. □

Theorem 3.5.8. ([AlThag95], p. 615) *Let X be a strong WM-*starshaped *metric space, $T : X \to X$, $K \subseteq X$, and $p \in M$ a fixed point of T. If $P_K(p)$ is compact and q-starshaped, $T(\partial K \cap K) \subseteq K$, and T is $d(p, K)$-nonexpansive on $P_K(p) \cup \{p\}$, then T has a fixed point in $P_K(p)$.*

Proof. Let $y \in P_K(p)$. Then

$$y \in \partial K \cap K \quad \text{(by Lemma 3.5.6)}.$$

Since $T(\partial K \cap K) \subseteq K$, then $Ty \in K$. by the $d(p, K)$-nonexpansiveness of T on $P_K(p) \cup \{p\}$,

$$d(Ty, p) = d(Ty, Tp) \leq d(y, p) \text{ with } d(y, p) \ \leq d(p, K).$$

So $d(Ty, p) \leq d(p, K)$. But it is impossible that $d(Ty, p) < d(p, K)$ (since $Ty \in K$). Then

$$d(Ty, p) = d(p, K).$$

Thus $Ty \in P_K(p)$.Therefore $T : P_K(p) \to P_K(p)$. Now, Theorem 3.5.2. with $D = P_K(p)$, and Lemma 3.5.5 show that T has a fixed point in $P_K(p)$. □

Corollary 3.5.9. ([AlThag95], p. 616) *Let X be a strong WCM space, $T : X \to X$, $K \subseteq X$, and $p \in X$ a fixed point of T. If $P_K(p)$ is compact and starshaped, $T(\partial K \cap K) \subseteq K$, and T is $d(p, K)$-nonexpansive on $P_K(p) \cup \{p\}$, then T has a fixed point in $P_K(p)$.*

Proof. If X is a strong convex metric space, then it is strong X-starshaped. Thus immediately, we can apply Theorem 3.5.8. □

3.6 Convergence of Iterations to Fixed Points in WCM Spaces

In this section, we present some results of Ciric-Ume-Khan [CUK03] and D. Ariza-Ruiza [ArRu12]. Motivated by Dotson's example [Do70], he considered a certain class of mappings which includes the classes of mappings studied by Zamfirescu, Ciric, Berinde and others and presented several results about convergence of distinct iterative processes in WCM spaces.

We recall: Let D be a nonempty subset of a metric space (X, d). A mapping $T : D \to X$ is said to be **contraction** if there exists a constant $\alpha \in [0, 1)$ such that,

$$d(Tx, Ty) \leq \alpha d(x, y). \tag{C}$$

for all $x, y \in D$. The well known Banach's fixed point theorem asserts that if $D = X$, T is contraction and (X, d) is complete, then T has a unique fixed point p in X, and for any $x_0 \in X$ the sequence $\{T^n(x_0)\}$ converges to p. This result has been extended by several authors to some classes of mappings by changing the contractive condition (C). For instance, two conditions that can replace (C) in Banach's theorem are the following:

(I) (Kannan, [Kan68]) There exists $\beta \in [0, \frac{1}{2})$ such that, for all $x, y \in D$,

$$d(Tx, Ty) \leq \beta[d(x, Tx) + d(y, Ty)]. \tag{K}$$

(II) (Chatterjea [Cha72]) There exists $\gamma \in [0, \frac{1}{2})$ such that, for all $x, y \in D$,

$$d(Tx, Ty) \leq \gamma[d(x, Ty) + d(y, Tx)]. \tag{Ch}$$

The conditions (C), (K) and (Ch) are independent (see [ArJiLo11], [Rh77] and [CS97]).

In 1972, Zamfirescu [Zam72], combining the conditions (C), (K) and (Ch), obtained a fixed point theorem for the class of mappings $T : X \to X$ for which there exists $\xi \in [0, 1)$ such that

$$d(Tx, Ty) \leq \xi. \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], \frac{1}{2}[d(x, Ty) + d(y, Tx)\}. \tag{Z}$$

A mapping satisfying (Z) is commonly called a Zamfirescu mapping. Note that the class of Zamfirescu mappings is a subclass of the class of mappings T satisfying the following condition: there exists $0 \leq h < 1$ such that

$$d(Tx, Ty) \leq h. \max\{d(x, y), \frac{1}{2}[d(x, Tx) + d(y, Ty)], d(x, Ty), d(y, Tx)\}. \tag{R}$$

This condition was first considered by Ciric [Cir71] who obtained a fixed point theorem for mappings satisfying (R). Recently, this class of mappings has been studied by Rafiq [Ra06]. Notice that every mapping T satisfying (R) is a quasicontraction. The concept of quasicontraction was introduced and investigated by Ciric [Cir71] in 1971, who obtained an existence fixed point theorem under the following condition: there exists a constant $k \in [0, 1)$ such that

$$d(Tx, Ty) \leq k. \max\{d(x, y), d(x, Tx), d(y, Ty), d(x, Ty), d(y, Tx)\}. \tag{QC}$$

Recently, Berinde [Ber04] proved a fixed point result using a new condition, which is independent of (QC). This condition can be stated as follows: there exist two constants $\theta \in [0, 1)$and $L \geq 0$ such that

$$d(Tx, Ty) \leq \theta.d(x, y) + L.d(y, Tx) \tag{B}$$

for all $x, y \in D$. To obtain the uniqueness of the fixed point of a mapping satisfying (B), Berinde considered the following contractive condition, quite similar to (B). There exist two constants $\delta \in [0, 1)$ and $L_1 \geq 0$ such that

$$d(Tx, Ty) \leq \delta.d(x, y) + L_1.d(x, Tx) \tag{B'}$$

for all $x, y \in D$.

Berinde noticed that the identity mapping on any metric space satisfies (B) but does not (B'). With the following example, we prove that both classes of mappings are independent.

Example 1. Let $X = \{a, b\}$ be any set together with the discrete metric d. The mapping $T : X \to X$,given by $Ta = b$ and $Tb = a$, satisfies (B') with $\delta \in (0, 1)$ arbitrary and $L_1 \geq 1 - \delta$. Indeed,

$$d(Ta, Tb) = 1 \leq \delta + L_1 = \delta d(a, b) + L_1 d(a, b) = \delta d(a, b) + L_1 d(a, Ta);$$

similarly,

$$d(Tb, Ta) = 1 \leq \delta + L_1 = \delta d(b, a) + L_1 d(b, a) = \delta d(b, a) + L_1 d(b, Tb).$$

Moreover, T does not satisfy (B), since if there exist $\theta \in [0, 1)$ and $L \geq 0$ such that T verifies (B), then

$$d(Ta, Tb) \leq \theta d(a, b) + Ld(b, Ta).$$

that is, $1 \leq \theta$, which is a contradiction.

Recall that: A W-convex metric space $(X, d, \oplus) = (X, d, W)$ is a metric space (X, d) together with a convexity mapping $W = \oplus : X \times X \times [0, 1] \to X$ satisfying

$$\begin{aligned} d(z, W(x, y, t)) &\leq td(z, x) + (1 - t)d(z, y) \\ d(z, tx \oplus (1 - t)y) &\leq td(z, x) + (1 - t)d(z, y), \end{aligned}$$

for all $x, y, z \in X, t \in [0, 1]$. Here we denote $W(x, y, \lambda) = tx \oplus (1 - t)y$

Example 2. Obviously, any normed space is a WCM space. Spaces of hyperbolic type, which were introduced by Goebel and Kirk [GK83] are convex metric spaces. As we shall see later, Hyperbolic spaces in the sense of Koolenbach [Koh05] are also WCM spaces; other examples of WCM spaces are CAT(0)-spaces and $\mathbb{R}$-trees (we will see that later on).

Every WCM space satisfies the following property, which is very important.

Theorem 3.6.1. ([ArRu12], p. 95) *If* $(X, d, \oplus)$ *is a WCM space, then*

$$\begin{aligned} d(x, W(x, y, t)) &= (1 - t)d(x, y) \text{ and } d(y, W(x, y, \lambda)) = td(x, y) \\ \text{or } d(x, tx \oplus (1 - t)y) &= (1 - t)d(x, y) \text{ and } d(y, tx \oplus (1 - t)y) = td(x, y) \end{aligned}$$

for all $x, y, \in X$ *and* $t \in [0, 1]$.

Recall that:

$$\begin{aligned} W(x, y, 0) &= 0x \oplus 1y = y, \\ W(x, y, 1) &= 1x \oplus 0y = x \\ W(x, x, t) &= tx \oplus (1-t)x = x \end{aligned}$$

A nonempty subset C of a W-convex metric space $(X, d, \oplus)$ is said to be W-convex if $tx \oplus (1-t)y \in C$ for all $x, y \in C$ and $\lambda \in [0, 1]$. A nice feature of our setting is that any W-convex subset is itself a W-convex metric space with the restriction of d and $\oplus$ to C.

Recall some iterative processes: Let D be a nonempty subset of a metric space (X, d) and $T : D \to D$ a self-mapping. Let $x_0 \in D$ be fixed, we can consider the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by

$$x_{n+1} := T(x_n) = T^{n+1}(x_0), \quad \text{for all } n \geq 0 \tag{1}$$

The sequence defined by (1) is known as the Picard iteration.

Let C be a closed subset of a complete metric space (X, d). If $T : C \to C$ satisfies any of the conditions (C), (K), (Ch), (Z), (R), (QC), (B), then T has at least a fixed point. Moreover, the Picard iteration converges to a fixed point of T. However, if any of this conditions is slightly weaker, then the Picard iteration need not converge to a fixed point of the operator T. The following trivial example shows this behavior.

Example. Let $X = \{x_1, x_2, x_3\}$, with the discrete metric d, and $T : X \to X$ defined by$T(x_1) = x_3, T(x_2) = x_2$ and $T(x_3) = x_1$. It is easy to check that T is nonexpansive, that is, $d(Tx_i, Tx_j) \leq d(x_i, x_j)$ for all $i, j \in \{1, 2, 3\}$. Moreover, the Picard iteration of T, with the starting point x_1 or x_3, does not converge to x_2, which is the fixed point of T.

In view of above example, we can state that some other iteration processes must be considered. Bearing in mind the iterative processes that exist in the Banach space setting, we shall introduce the most important iterative processes in the convex metric spaces. In order to do this, C will be a convex subset of a convex metric space $(X, d, \oplus)$ and $T : C \to C$ a mapping.

For any given x_0 in C, the sequence $\{x_n\}_{n=0}^{\infty}$ defined by

$$x_{n+1} = W(x_n, Tx_n, \lambda) = \lambda x_n \oplus (1-\lambda)Tx_n, \qquad \text{for all } n \geq 0. \tag{2}$$

where $\lambda \in [0, 1]$, is called Krasnosel'skij iteration [Kr55].

Mann iteration [Ma53] is essentially an averaged algorithm which generates a sequence recursively

$$x_{n+1} = W(x_n, Tx_n, \alpha_n) = \alpha_n x_n \oplus (1-\alpha_n)Tx_n, \qquad \text{for all } n \geq 0. \tag{3}$$

where the initial guess $x_0 \in C$ and $\{\alpha_n\}_{n\geq 0}$ is a sequence in $[0,1]$.

Ishikawa iteration [Ish76] is the following process of two steps: let $x_0 \in X$ be fixed, consider the sequence $\{x_n\}_{n\geq 0}$ defined by

$$\begin{cases} y_n = W(x_n, Tx_n, \beta_n) = \beta_n x_n \oplus (1-\beta_n)Tx_n, \\ x_{n+1} = W(x_n, Ty_n, \alpha_n) = \alpha_n x_n \oplus (1-\alpha_n)Ty_n, \quad \text{for all } \ n \geq 0, \end{cases} \tag{4}$$

where $\{\alpha_n\}_{n\geq 0}$ and $\{\beta_n\}_{n\geq 0}$ are sequence in $[0,1]$.

The class of φ-quasinonexpansive mappings

Example. (Dotson [Do70]) Consider the following self-mapping $T : \mathbb{R} \to \mathbb{R}$, defined by

$$Tx = \begin{cases} \frac{x}{2}\sin(\frac{1}{x}) & \text{if } \ x \neq 0, \\ 0 & \text{if } \ x = 0. \end{cases}$$

Then T satisfies the property

$$d(Tx,p) \leq \frac{1}{2} d(x,p),$$

for all $x \in \mathbb{R}$, $p \in Fix(T) = \{0\}$.

Solution. Indeed, since $p = 0$,if $x \neq 0$,

$$\begin{aligned} d(Tx,0) &= |Tx-0| = |Tx| = \left|\frac{x}{2}\sin(\frac{1}{x})\right| \\ &= \frac{1}{2}\left|x\sin(\frac{1}{x})\right| = \frac{1}{2}|x|\left|\sin(\frac{1}{x})\right| \\ &\leq \frac{1}{2}|x| = \frac{1}{2}d(x,0). \quad \square \end{aligned}$$

Thus, motivated by this example, we can consider the following class of mappings.

Definition: Let D be a nonempty subset of a metric space (X,d). We say that $T : D \to X$ is a **φ-quasinonexpansive mapping** if $Fix(T) \neq \varnothing$ and there exists a function $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that

$$d(Tx,p) \leq \varphi(d(x,p)). \tag{5}$$

for all $x \in X$, $p \in Fix(T)$.

Remarks. (1) Recently, Olatinwo have studied intensively contractive conditions that includes as a particular case the class of φ-nonexpansive mappings (see [Ol08], [Ol11]) and many other papers by the same author).

(2) Notice that if we take φ as the identity function, we obtain the concept of quasi-nonexpansiveness, which was introduced by Tricomi [Tri16] for real functions and later studied by Diaz and Metcalf [DM67], [DM69] and by Dotson [Do70] for mappings in Banach spaces (see [Ber07, Section 3.5, Section 4.2] for detailed discussion of this and related notions.).

(3) One can easily show that every contraction mapping is a φ-quasinonexpansive mapping, with $\varphi(t) = \alpha t$ for $t \in \mathbb{R}^+$, using Banach's fixed point theorem.

(4) However, Dotson's example shows that the class of φ-quasinonexpansive mappings properly includes contractive mappings.

(5) Next, we now proceed to show that the contractive mappings from previous subsection are in the class of φ-quasinonexpansive mappings. Moreover, this example can be generalized in such a way that the resulting map is not contractive.

Example 1. Let $\varphi : \mathbb{R}^+ \cup \{0\} \to \mathbb{R}^+ \cup \{0\}$ be a function such that $\varphi(t) < t$ for each $t > 0$. Let X be the real line with the usual metric. The mapping $T : \mathbb{R} \to \mathbb{R}$ defined by

$$Tx = \begin{cases} \varphi(x)\sin(\frac{1}{x}) & \text{if } x \neq 0, \\ 0 & \text{if } x = 0. \end{cases}$$

is a φ-quasinonexpansive mapping but T is not a contraction mapping.

Example 2. ([Ber07], p. 39) (i) Any Kannan mappings $T : X \to X$ is a φ-quasinonexpansive mappings with

$$\varphi(t) := \frac{\beta}{1-\beta} t \text{ for all } t \in \mathbb{R}^+, \quad \beta \in [0, \frac{1}{2}).$$

(ii) Any Chatterjea mapping $T : X \to X$ is a φ-quasinonexpansive mapping, with

$$\varphi(t) = \frac{\gamma}{1-\gamma} t \text{ for all } t \in \mathbb{R}^+, \quad \gamma \in [0, \frac{1}{2}).$$

Solution. (i) If T is a Kannan mapping. Then from (K) with $y = p \in Fix(T)$ we get

$$\begin{aligned} d(Tx,p) &\leq \beta[d(x,Tx) + d(p,Tp)] = \beta d(x,Tx) \leq \beta[d(x,p) + d(p,Tx)] \\ &\leq \beta[d(x,p) + \beta[d(p,Tp) + d(x,Tx)]] = \beta[d(x,p) + \beta d(x,Tx)] \\ &\leq \beta[d(x,p) + \beta[d(x,p) + d(p,Tx)]] \\ &= \beta[d(x,p) + \beta d(x,p) + \beta d(p,Tx)] \\ &\leq \beta[d(x,p) + \beta d(x,p) + \beta[\beta[d(p,Tp) + d(x,Tx)]] \\ &= \beta[d(x,p) + \beta d(x,p) + \beta\beta d(x,Tx)] \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq \beta d(x,p) + \beta^2 d(x,p) + \beta^3 d(x,p) + \ldots (\text{geometric series }) \\ &= \frac{\beta}{1-\beta} d(x,p) \end{aligned}$$

(ii) Similar to part (i). $\square$

The following result, which is implicitly included in [Ber07], shows that this fact is still true for a more general class of mappings.

Theorem 3.6.2. [Ber07] *Let (X, d) be a complete metric space. If $T : X \to X$ satisfies* (QC)*, then T is a φ-quasinonexpansive mapping, with* $\varphi(t) := \max\{k, \frac{k}{1-k}\}t$.

Proof. Ciric [Cir74] proved that T has a unique fixed point p in X. Taking $y = p$ in (QC) we get

$$\begin{aligned} d(Tx, p) &= d(Tx, Tp) \\ &\leq k \ \max\{d(x, p), d(x, Tx), d(p, Tp), d(x, Tp), d(p, Tx)\} \\ &\leq k \ \max\{d(x, p), d(x, p) + d(p, Tx), d(p, Tx)\}. \end{aligned}$$

for each x in X. Since $0 \leq k < 1$, it is impossible that $d(Tx, p) \leq k \ d(p, Tx)$. Thus

$$d(Tx, p) \leq k \ \max\{d(x, p), d(x, p) + d(p, Tx)\}.$$

If $\max\{d(x, p), d(x, p) + d(p, Tx)\} = d(x, p) + d(p, Tx)$, then

$$\begin{aligned} d(Tx, p) &\leq k[d(x, p) + d(p, Tx)] \\ &\leq k[d(x, p) + k[d(x, p) + d(p, Tx)] \\ &\leq k[d(x, p) + kd(x, p) + k[d(x, p) + d(p, Tx)] \\ &\leq \ldots\ldots\ldots\ldots\ldots\ldots \\ &\leq kd(x, p) + k^2 d(x, p) + k^3 d(x, p) + \ldots. \\ &= \frac{k}{1-k} d(x, p) \end{aligned}$$

Thus we deduce

$$d(Tx, p) \leq \max\{k, \frac{k}{1-k}\} d(x, p),$$

for every $x \in X$. □

In the case of mappings satisfying (R), we can obtain a better function φ.

Theorem 3.6.3. [Ber07] *Let (X, d) be a complete metric space. If $T : X \to X$ satisfies* (R)*, then T is a φ-quasinonexpansive mapping, with $\varphi(t) := ht$.*

Proof. By Ciric [Cir74], we know that T has a unique fixed point in X, say p. If we take $y = p$ in (R) we get

$$\begin{aligned} d(Tx, p) &= d(Tx, Tp) \\ &\leq h. \max\{d(x, p), \frac{1}{2}[d(x, Tx) + d(p, Tp)], d(x, Tp), d(p, Tx)\} \\ &\leq h. \max\{d(x, p), \frac{1}{2}[d(x, p) + d(p, Tx)], d(p, Tx)\} \end{aligned}$$

for each x in X.Since $0 \leq h < 1$, it is impossible that $d(Tx, p) \leq h.d(p, Tx)$. Thus

$$d(Tx, p) \leq h. \max\{d(x, p), \frac{1}{2}[d(x, p) + d(p, Tx)],$$

Using a similar argument in Theorem 3.6.2 we get

$$d(Tx,p) \leq \max\{h, \frac{h}{2-h}\}d(x,p) = hd(x,p),$$

for every $x \in X$. □

A trivial verification shows that if T has at least one fixed point and satisfies (B'), then T is a φ-quasi-nonexpansive mapping, with $\varphi(t) = \delta t$ for each $t \in \mathbb{R}^+$.

We must notice that there exist other classes of mappings which belong to the class of φ-quasi-nonexpansive mappings. For example, in [Jag77].

Convergence results

Here and subsequently, Φ denotes the family of functions $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ such that φ is continuous and $\varphi(t) < t$ for all $t > 0$. Before we discuss our results, we state an elementary numerical result.

Lemma 3.6.4. ([ArRu12], p. 98) *Let $\{\lambda_n\}_{n\in\mathbb{N}}$ be a real sequence in $[0,1]$ and let $\{d_n\}_{n\in\mathbb{N}}$ be a sequence of nonnegative real numbers such that*

$$d_{n+1} \leq \lambda_n d_n + (1-\lambda_n)\varphi(d_n) \quad \text{for all } n \in \mathbb{N}, \tag{6}$$

where $\varphi \in \Phi$. If $\{\lambda_n\}_{n\in\mathbb{N}}$ converges to $\lambda \in (0,1]$, then we have $\lim\limits_{n\to\infty} d_n = 0$.

Proof. Since $\varphi(t) < t$ for all $t \in \mathbb{R}^+$, we get that $\{d_n\}_{n\in\mathbb{N}}$ is nonincreasing since

$$d_{n+1} \leq \lambda_n d_n + (1-\lambda_n)\varphi(d_n) < \lambda_n d_n + (1-\lambda_n)d_n = d_n$$

and, therefore, convergent to a nonnegative real number d. We shall show that $d = 0$. In order to do this, we assume that $d > 0$ and we obtain a contradiction as follows. Since $\lambda_n \to \lambda \in (0,1]$, as $n \to \infty$, taking limits in (6) we have that

$$d \leq \lambda d + (1-\lambda)\varphi(d) < \lambda d + (1-\lambda)d = d,$$

which is a contradiction. Therefore, $d = 0$, that is, $\lim\limits_{n\to\infty} d_n = 0$. □

We now prove the convergence of the Mann iteration process on a WCM space, when the operator T is assumed to be only φ-quasinonexpansive.

Theorem 3.6.5. ([ArRu12], p. 98) *Let C be a convex subset C of a WCM space. Assume that $T : C \to C$ is a φ-quasinonexpansive mapping with $\varphi \in \Phi$. Let $\{\alpha_n\}_{n\in\mathbb{N}}$ be a real sequence in $[0,1]$ such that $\{\alpha_n\}_{n\in\mathbb{N}}$ converges to some positive real number. Then, for any x_0 in X, the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (3) converges to the unique fixed point of T.*

Proof. Let us first prove that T has a unique fixed point in C. Suppose that $p, q \in Fix(T)$, with $p \neq q$. Using (5) and the property of φ, we obtain

$$d(q,p) = d(Tq,p) \leq \varphi(d(q,p)) < d(q,p).$$

which is a contradiction. Let $x_0 \in C$ be arbitrary. Now, we shall prove that the Mann iteration $\{x_n\}_{n\in\mathbb{N}}$ converges to p, where $Fix(T) = \{p\}$. Since

$$\begin{aligned}
d(x_{n+1}, p) &= d(W(x_n, Tx_n, \alpha_n), p) \\
&= d(\alpha_n x_n \oplus (1-\alpha_n)Tx_n, p) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n) d(Tx_n, p) \\
&\leq \alpha_n d(x_n, p) + (1-\alpha_n)\varphi(d(x_n, p)) \ (T \text{ is } \varphi\text{-quasi-NE}).
\end{aligned}$$

for every $n \in \mathbb{N}$, by Lemma 3.6.4 $\lim\limits_{n\to\infty} d(x_{n+1}, p) = 0$ so we deduce that $\{x_n\}_{n\in\mathbb{N}}$ converges to p. □

Remark. Clearly, if we take $\{\alpha_n\}_{n\in\mathbb{N}}$ as a constant sequence in $[0,1]$, we get a result about the convergence of Krasnosel'skij iteration.

Notice that if in the above result we take $\alpha_n = 0$ for all $n \geq 0$, we obtain a result about the convergence of the Picard iteration process. Moreover, this result still holds if it is just assumed that C is a nonempty subset of a metric space (X, d).

Corollary 3.6.6. ([ArRu12], p. 99) *Let C be a nonempty subset of a metric space (X, d). If $T : C \to C$ is a φ-quasinonexpansive mapping, with $\varphi : \mathbb{R}^+ \to \mathbb{R}^+$ being a continuous function such that $\varphi(t) < t$ for all $t > 0$, then the sequence $\{x_n\}_{n\in\mathbb{N}}$ defined by (1) converges to the unique fixed point of T, for any x_0 in X.*

Now, we present some results of Ciric-Ume-Khan [CUK03] on the convergence of Ishikawa iterates.

In 1983, Naimpally and Singh [NS83] have studied the mappings which satisfy the contractive definition introduced in [Cir77]. They proved the following:

Theorem 3.6.7. [NS83] *Let X be a normed vector space and C be a nonempty closed convex subset of X. Let $T : C \to C$ be a selfmapping satisfying*

$$\|Tx - Ty\| \leq h.\max\{\|x-y\|, \|x-Tx\|, \|y-Ty\|, \|x-Ty\| + \|y-Tx\|\} \tag{A}$$

for all x, y in C, where $0 \leq h < 1$ and let $\{x_n\}$ be the sequence of the Ishikawa-scheme associated with T , that is, $x_0 \in C$,

$$\begin{aligned}
y_n &= (1-\beta_n)x_n + \beta_n Tx_n, \quad n \geq 0, \\
x_{n+1} &= (1-\alpha_n)x_n + \alpha_n Ty_n, \quad n \geq 0,
\end{aligned}$$

where $0 \leq \alpha_n$, $\beta_n \leq 1$. If $\{\alpha_n\}$ is bounded away from zero and if $\{x_n\}$ converges to p, then p is a fixed point of T .

In [CUK03] generalizing of the result of Naimpally and Singh to a pair of mappings S and T , defined on a WCM , which satisfy the following condition:

$$d(Sx, Ty) \leq h[d(x, y) + d(x, Ty) + d(y, Sx)], \tag{B}$$

where $0 < h < 1$.

It is clear that the condition (B) is very general, since by the triangle inequality, condition (B) is always satisfied with $h = 1$.

Theorem 3.6.8. ([CUK03], p. 124) *Let C be a nonempty closed convex subset of a WCM space X and let $S, T : C \to C$ be self-mappings satisfying (B) for all x, y in C. Suppose that $\{x_n\}$ is Ishikawa type iterative scheme associated with S and T, defined by*

$$\begin{aligned} x_0 &\in C, && (1)\\ y_n &= W(Sx_n, x_n, \beta_n), \quad n \geq 0 && (2)\\ x_{n+1} &= W(Ty_n, x_n, \alpha_n), \quad n \geq 0 && (3) \end{aligned}$$

where $\{\alpha_n\}$ and $\{\beta_n\}$ satisfy $0 \leq \alpha_n$, $\beta_n \leq 1$ and $\{\alpha_n\}$ is bounded away from zero. If $\{x_n\}$ converges to some point $p \in C$, then p is the common fixed point of S and T.

Proof. From properties of convex structure, we have:

$$d(x, W(x, y, t)) = (1 - t)d(x, y); \quad d(y, W(x, y, t)) = td(x, y).$$

From (3) it follows that

$$d(x_n, x_{n+1}) = d(x_n, W(Ty_n, x_n, \alpha_n)) = \alpha_n d(x_n, Ty_n).$$

Since $x_n \to p$ as $n \to \infty$, $d(x_n, x_{n+1}) \to 0$ as $n \to \infty$. Since $\{\alpha_n\}$ is bounded away from zero, it follows that

$$\lim_{n\to\infty} d(x_n, Ty_n) = 0. \tag{4}$$

From (2) we have

$$\begin{aligned} d(x_n, y_n) &= d(x_n, W(Sx_n, x_n, \beta_n)) = \beta_n d(x_n, Sx_n),\\ d(Sx_n, y_n) &= d(Sx_n, W(Sx_n, x_n, \beta_n)) = (1 - \beta_n)d(x_n, Sx_n). \end{aligned}$$

Thus we have

$$\begin{aligned} d(Sx_n, Ty_n) &\leq h[d(x_n, y_n) + d(x_n, Ty_n) + d(y_n, Sx_n)] \quad \text{(using (B))}\\ &= h[\beta_n d(x_n, Sx_n) + d(x_n, Ty_n) + (1 - \beta_n)d(x_n, Sx_n)]\\ &= h[d(x_n, Sx_n) + d(x_n, Ty_n)]. \end{aligned}$$

Now

$$\begin{aligned} d(Sx_n, Ty_n) &\leq h[d(x_n, Sx_n) + d(x_n, Ty_n)]\\ &\leq h[d(Sx_n, Ty_n) + d(x_n, Ty_n) + d(x_n, Ty_n)]\\ \text{or } d(Sx_n, Ty_n) - hd(Sx_n, Ty_n) &\leq 2hd(x_n, Ty_n). \end{aligned}$$

Hence

$$d(Sx_n, Ty_n) \leq \frac{2h}{1-h} d(x_n, Ty_n) \quad (h < 1).$$

Taking the limit as $n \to \infty$, we obtain, by (4),

$$\lim_{n\to\infty} d(Sx_n, Ty_n) = 0.$$

Since $Ty_n \to p$ as $n \to \infty$ (by 4), it follows that $Sx_n \to p$ as $n \to \infty$. Since

$$d(x_n, y_n) = \beta_n d(x_n, Sx_n) \leq d(x_n, Sx_n) \quad (\beta_n \leq 1)$$

and

$$\lim_{n\to\infty} d(x_n, Sx_n) = 0,$$

it follows also that $y_n \to p$ as $n \to \infty$.

From (B) again, we have

$$d(Sx_n, Tp) \leq h[d(x_n, p) + d(x_n, Tp) + d(p, Sx_n)].$$

Taking the limit as $n \to \infty$, we obtain

$$d(p, Tp) \leq hd(p, Tp).$$

Since $h < 1$, $d(p, Tp) = 0$. Hence $Tp = p$. Similarly, from (B),

$$d(Sp, Ty_n) \leq h[d(p, y_n) + d(p, Ty_n) + d(y_n, Sp)].$$

Taking the limit as $n \to \infty$, we get

$$d(Sp, p) \leq hd(p, Sp).$$

Hence $Sp = p$. Therefore, $Sp = Tp = p$ and the proof is complete. □

Corollary 3.6.9. ([CUK03], p. 126) *Let X be a normed vector space and C be a closed convex subset of X. Let $S, T : C \to C$ be two mappings satisfying (B) and $\{x_n\}$ be the sequence of Ishikawa-scheme associated with S and T ; for $x_0 \in C$,*

$$\begin{aligned} y_n &= (1-\beta_n)x_n + \beta_n T x_n, \quad n \geq 0, \\ x_{n+1} &= (1-\alpha_n)x_n + \alpha_n T y_n, \quad n \geq 0. \end{aligned}$$

If $\{\alpha_n\}$ is bounded away from zero and $\{x_n\}$ converges to p, then p is a common fixed point of S and T. □

Chapter 4

M-convex and M^*-convex Metric Spaces

In this chapter we present a study of different types of convexities such as M-convex and M^*-convex in metric spaces and relationship between them. For these spaces we include strict convexity. Furthermore, we consider its applications in fixed point theory and Best approximation.

4.1 Relationship between various types of convex metric spaces

Let (X, d) be a metric space. In this section we discuss different definitions of convexities in (X, d) and their equivalence. These types are M-convex, M^*-convex, M^{**}-convex and M^{***}-convex (due to Khalil [Khal88] and Khamsi (communicated)).

Definition: (X, d) is called M**-convex** if for any $x, y \in X$, with $x \neq y$, there exists $z \in X$ such that

$$d(x, y) = d(x, z) + d(y, z).$$

Definition: (X, d) is called M^***-convex** if for any $x, y \in X$, with $x \neq y$, and $t \in [0, 1]$, there exists $z = z_t \in X$ (different from x and y) such that

$$d(x, z) = td(x, y) \text{ and } d(y, z) = (1 - t)d(x, y). \tag{I}$$

Lemma 4.1.1. [Kham12] *(X, d) is M^*-convex $\Rightarrow$ (X, d) is M-convex.*
Proof. Indeed, by (I), we have

$$d(x, y) = td(x, y) + (1 - t)d(x, y) = d(x, z_t) + d(y, z_t).]$$

Note. The converse is not true, i.e. (X, d) is M-convex $\not\Rightarrow$ (X, d) is M^*-convex.

Examples. Let $X = Q$, the set of rational numbers. Then Q is M-convex. [Indeed, for $x, y \in Q$, $x \neq y$, set $z = \frac{x+y}{2} \in Q$, then we have

$$\begin{aligned} d(x,z) &= |x - \frac{x+y}{2}| = \frac{1}{2}|x-y| = \frac{1}{2}d(x,y), \\ \text{and } d(y,z) &= |y - \frac{x+y}{2}| = \frac{1}{2}|y-x| = \frac{1}{2}d(x,y); \end{aligned}$$

i.e.,

$$d(x,z) = \frac{1}{2}d(x,y) \text{ and } d(y,z) = \frac{1}{2}d(x,y).$$

In particular, we have $d(x,y) = d(x,z) + d(z,y)$. So Q is M-convex.]

But Q is not M^*-convex. [Indeed, set $x = 0$ and $y = 1$. Then, for any $t \in [0,1]$ which is irrational, and any $z \in Q$,

$$d(x,z) = |x-z| = |z| \in Q, \text{ while } td(x,y) = t|x-y| = t \notin Q,$$

and so $d(x,z) \neq td(x,y)$. Also

$$d(y,z) = |y-z| = |1-z| \in Q, \text{ while } (1-t)d(x,y) = (1-t)|x-y| = 1-t \notin Q,$$

and so $d(y,z) \neq (1-t)d(x,y)$. Hence there does not exist $z = z_t \in Q$ such that

$$d(x,z_t) = td(x,y) \text{ and } d(y,z_t) = (1-t)d(x,y);$$

i.e. (X,d) is not M^*-convex.] $\Box$

In fact, this example suggests the following definition which is very commonly used.

Definition: (X,d) is called **half-convex** if for any $x, y \in X$, with $x \neq y$, there exists $z \in X$ such that

$$d(x,z) = \frac{1}{2}d(x,y) \text{ and } d(y,z) = \frac{1}{2}d(x,y).$$

We will write $z = \frac{x \oplus y}{2}$.

Theorem 4.1.2. [Kham12] *Let (X,d) be a half-convex metric space. Suppose X is complete. Then (X,d) is M-convex. More precisely, for any $x, y \in X$, the interval*

$$[x,y] = \{tx \oplus (1-t)y : t \in [0,1]\}$$

is contained in X. Further, the segment $[x,y]$ is isometric to the real interval $[0, d(x,y)]$ and preserves the convexity properties of the points in $[0, d(x,y)]$.

Proof. Let $x, y \in X$ with $x \neq y$. Let us introduce a **new notation:**

$$\begin{aligned} z(1,2) &= \frac{x \oplus y}{2} \text{ means} \\ d(x, z(1,2)) &= \frac{1}{2}d(x,y) \qquad (1) \\ \text{and } d(y, z(1,2)) &= \frac{1}{2}d(x,y) \qquad (2) \end{aligned}$$

If we keep going (having mid-points or bisecting), we get

$$\begin{aligned} z(1,4) &= {}^{Def} \frac{x \oplus z(1,2)}{2}, \\ z(2,4) &= {}^{Def} z(1,2) \\ \text{and } z(3,4) &= {}^{Def} \frac{z(1,2) \oplus y}{2}. \end{aligned}$$

It is easy to check using the triangle inequality that

$$\begin{aligned} d(x, z(k,4)) &= \frac{k}{4} d(x,y), \\ d(y, z(k,4)) &= (1 - \frac{k}{4}) d(x,y), \\ \text{and } d(z(k,4), z(s,4)) &= |\frac{k}{4} - \frac{s}{4}| d(x,y), \end{aligned}$$

for any $k, s \in \{1,2,3\}$.

[Explanation for $k, s = 1,2,3$:

$$\begin{aligned} z(1,4) &= \frac{x \oplus z(1,2)}{2} = \text{ means} \\ d(x, z(1,4)) &= \frac{1}{2} d(x, z(1,2)) =^{(1)} \frac{1}{2}.[\frac{1}{2} d(x,y)] = \frac{1}{4} d(x,y) \quad (3) \\ \text{and } d(z(1,2), z(1,4)) &= \frac{1}{2} d(x, z(1,2)) = \frac{1}{2}[\frac{1}{2}(d(x,y)] = \frac{1}{4}(d(x,y); \quad (4) \\ \text{or } d(z(2,4), z(1,4)) &= |\frac{2}{4} - \frac{1}{4}| d(x,y)], \\ \text{also } d(z(1,2), z(1,4)) &\leq d(z(1,2), x) + d(x, z(1,4)), \\ &= \frac{1}{2} d(x,y) + \frac{1}{4} d(x,y), \text{ by (1) and (3)} \\ &= \frac{3}{4}(d(x,y) = (1 - \frac{1}{4}) d(x,y), \quad (5) \end{aligned}$$

Next,

$$\begin{aligned} z(3,4) &= \frac{z(1,2) \oplus y}{2} \text{ means} \\ d(y, z(3,4)) &= \frac{1}{2} d(y, z(1,2)) =^{(2)} \frac{1}{2}.[\frac{1}{2} d(x,y)] = \frac{1}{4} d(x,y), \quad (6) \\ \text{and } d(z(1,2), z(3,4)) &= \frac{1}{2} d(z(1,2), y) =^{(2)} \frac{1}{2}.[\frac{1}{2} d(x,y)] = \frac{1}{4} d(x,y); \quad (7) \\ \text{or } d(z(2,4), z(3,4)) &= |\frac{2}{4} - \frac{3}{4}| d(x,y)], \\ \text{also } d(z(1,2), z(3,4)) &\leq d(z(1,2), y) + d(y, z(3,4)) \\ &= \frac{1}{2} d(x,y) + \frac{1}{4} d(x,y), \text{ by (2) and (6)} \\ &= \frac{3}{4} d(x,y) = (1 - \frac{1}{4}) d(x,y) \quad (8)\ ] \end{aligned}$$

By induction, we will be able to construct a sequence of points $\{z(k,n)\} \in X$ with $k \in \{1, 2, ..., 2^n - 1\}$, for any $n \geq 1$ such that the map $T : K \to X$, where $K = \{\frac{k}{2^n} d(x,y) : k = 0, 1, ..., 2^n, \text{ and } n \geq 1\}$, defined by

$$T(0) = x, T(1) = y \text{ and } T(\frac{k}{2^n} d(x,y)) = z(k,n),$$

is an isometry, i.e. $d(T(t), T(s)) = |t - s|$, for any $t, s \in K$. In other words, we almost have a linear segment in X with end-points x and y. Assume moreover that X is complete. Then by density of K in the real interval $[0, d(x,y)]$, we get that there exists an interval in X with end points x and y, i.e. for any $t \in [0,1]$, there exists $z = t \oplus (1-t)y \in X$. Moreover the segment $[x,y]$ is isometric to the real interval $[0, d(x,y)]$ and preserves the convexity properties of the points in $[0, d(x,y)]$. $\Box$

Remark. If X fails to be complete, then this result is not true as the example $X = Q$ shows.

Definition: (X,d) is called M^{**}**-convex** if for any $x, y \in X$, with $x \neq y$, and $\alpha, \beta \in (0, +\infty)$, such that whenever $d(x,y) \leq \alpha + \beta$, we have

$$B[x,\alpha] \cap B[y,\beta] \neq \emptyset,$$

The last definition in this section is as follows.

Definition: (X,d) is called M^{***}**-convex** if for any $x, y \in X$, with $x \neq y$, and $\alpha \in [0, d(x,y)]$, i.e. $0 \leq \alpha \leq d(x,y)$, we have

$$B[x,\alpha] \cap B[y, d(x,y) - \alpha] \neq \emptyset.$$

In the next result we discuss the equivalence between the above convexity properties.

Theorem 4.1.3. [Kham12] *Let (X,d) be a metric space. Then the following statements are equivalent:*

(a) *(X,d) is M^*-convex.*

(b) *(X,d) is M^{**}-convex.*

(c) *(X,d) is M^{***}-convex.*

Proof. $(a) \Rightarrow (b)$: Assume that (X,d) is M^*-convex. Let $x, y \in X$, with $x \neq y$, and $\alpha, \beta \in (0, +\infty)$, such that whenever $d(x,y) \leq \alpha + \beta$. Set $t = \frac{\alpha}{\alpha+\beta} \in [0,1]$. Then we have $1 - t = \frac{\beta}{\alpha+\beta}$. Since (X,d) is M^*-convex, there exists $z \in X$ such that

$$d(x,z) = td(x,y) \text{ and } d(y,z) = (1-t)d(x,y).$$

Since $d(x,y) \leq \alpha + \beta$, we get

$$\begin{aligned} d(x,z) &= td(x,y) = \frac{\alpha}{\alpha+\beta} d(x,y) \leq \alpha, \\ d(y,z) &= (1-t)d(x,y) = \frac{\beta}{\alpha+\beta} d(x,y) \leq \beta. \end{aligned}$$

Hence $z \in B[x,\alpha] \cap B[y,\beta]$, i.e., $B[x,\alpha] \cap B[y,\beta] \neq \emptyset$.

$(b) \Rightarrow (c)$: Next assume (X,d) is M^{**}-convex. It is easy to see that (X,d) is M^{***}-convex. [Indeed let $x, y \in X$, with $x \neq y$, and $\alpha \in [0, d(x,y]$, i.e. $0 \leq \alpha \leq d(x,y)$. Set $\beta = d(x,y) - \alpha$. Then we have $d(x,y) \leq \alpha + \beta$. Since (X,d) is M^{**}-convex, we have $B[x,\alpha] \cap B[y,\beta] \neq \emptyset$, i.e.

$$B[x,\alpha] \cap B[y, d(x,y) - \alpha] \neq \emptyset.]$$

$(c) \Rightarrow (a)$: Finally assume (X,d) is M^{**}-convex. To prove that (X,d) is M^*-convex. let $x, y \in X$, with $x \neq y$, and $\alpha \in [0,1]$. Since $0 \leq \alpha d(x,y) \leq d(x,y)$, and (X,d) is M^{***}-convex, we get

$$B[x, \alpha d(x,y)] \cap B[y, d(x,y) - \alpha d(x,y)] \neq \emptyset.$$

Let $z \in B[x, \alpha d(x,y)] \cap B[y, d(x,y) - \alpha d(x,y)]$. Then we have

$$d(x,z) \leq \alpha d(x,y) \text{ and } d(y,z) \leq d(x,y) - \alpha d(x,y) = (1-\alpha) d(x,y). \tag{II}$$

Following an earlier argument on page 1, using (II),

$$d(x,y) \leq d(x,z)) + d(y,z) \leq d(x,z) + (1-\alpha) d(x,y),$$

or, $\alpha d(x,y) \leq d(x,z)$; hence $d(x,z) = \alpha d(x,y)$. Next, using (II) again,

$$d(x,y) \leq d(x,z) + d(y,z) \leq \alpha d(x,y) + d(y,z),$$

or, $(1-\alpha) d(x,y) \leq d(y,z)$; hence $d(y,z) = (1-\alpha) d(x,y)$. Thus

$$d(x,z) = \alpha d(x,y) \text{ and } d(y,z) = (1-\alpha) d(x,y).$$

i.e., (X,d) is M^*-convex. □

4.2 Strictly M^*-Convex Metric Spaces

In this section we study strictly M^*-convex metric space and strictly M^*-convex metric space with M^*-convex round balls, as given by Bula in [Bul99]. These objects generalize well known concept of strictly convex Banach space.

Remarks. (1) We can define convexity in the ordinary sense only in vector space.

(2) In the bibliography we find several possibilities as concept of convexity of vector space transfer to space with metric or topology.

(3) What properties of convexity are essential? From works of other authors, solid indications are two:

(i) Intersection of a convex sets is convex set;

(ii) Closed balls are convex sets.

(4) If we can guarantee these properties in considered space, we say **structure of convexity** is formed in space.

(5) The condition of convexity for definition set or range set of mapping very often is used for existence of fixed point of mapping in the theory of fixed points.

(6) Several mathematicians have attempted transfer of structure of convexity (such W-convexity, M-convexity, M^*-convexity) as to space which is not vector space.

It is known from Section 2.1 that, for any convex closed subset C of a strictly convex Banach space $(X, ||.||)$ and nonexpansive mapping $T : C \to C$, the set $Fix(T)$ is convex and closed. This property is also true for broader classes of mappings, for example, for quasi-nonexpansive and asymptotically nonexpansive mappings (see Section 2.1).

Recall that**:** A set $K \subseteq (X, d)$ is said to be M^***-convex** if for each $x, y \in K$ and for each $t \in [0, 1]$ there exists a $z \in K$ that satisfies:

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y).$$

But by means of previous definition closed balls may be non-convex sets and intersection of a convex sets may be non-convex set (see, for example, I. Galina [Ga92]). Therefore we define strictly M^*-convex metric space in following manner:

Definition: A metric space X is said to be **strictly** M^***-convex** if, for each $x, y \in X$ and for each $t \in [0, 1]$, there exists a unique $z \in X$ that satisfies:

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y). \tag{1}$$

Note. Every strictly convex Banach space $(X, ||.||)$ is strictly M^*-convex.

Proof. By a result of V.I. Istratescu (1981)Ist81], a Banach space $(X, ||.||)$ is strictly convex iff, for any $x, y \in X$ and $t \in [0, 1]$, $\exists$ a unique $z \in X$ such that

$$\|x - z\| = t \|x - y\|, \quad \|z - y\| = (1 - t) \|x - y\|.$$

Thus $(X, ||.||)$ is strictly M^*-convex. $\square$

Remark. It is simple to prove that intersection of M^*-convex sets is M^*-convex set in strictly M^*-convex metric space (I. Galina [Ga92]). But question of M^*-convexity of closed balls is still open.

Strictly M^*-convex metric space with M^*-convex round balls

Since we can not guarantee that closed balls in strictly M^*-convex metric space are M^*-convex sets, we require this condition in addition. We define:

Definition: A strictly M^*-convex metric space X is said to be **strictly M^*-convex space** with **M^*-convex round balls** if for any $a, b, c \in X$ $(a \neq b)$, $t \in]0,1[$, there exists a $z \in X$ such that

$$\begin{aligned} d(a,z) &= td(a,b) \text{ and } d(z,b) = (1-t)d(a,b), \\ d(c,z) &< \max\{d(c,a), d(c,b)\}. \end{aligned} \tag{2}$$

Lemma 4.2.1. ([Bul99], p. 8) *Let X be a strictly M^*-convex metric space with M^*-convex round balls. Then closed ball $B[c,r]$ for every $r > 0$ and every $c \in X$ is a M^*-convex set.*

Proof. We fix $r > 0$ and $c \in X$. We choose freely two points $a, b \in B[c,r]$, $a \neq b$, and fix $t \in]0,1[$.By definition of strictly M^*-convex metric space, there exists a unique $z \in X$ such that

$$d(a,z) = td(a,b) \text{ and } d(z,b) = (1-t)d(a,b).$$

We must prove that $z \in B[c,r]$ or $d(c,z) \leq r$. By condition of M^*-convex round balls follows that

$$d(c,z) < \max\{d(c,a), d(c,b)\} \leq r. \ \square$$

Remarks. (a) It can be proved that condition (2) above is equivalent with condition of M^*-convexity of closed balls. We require more. With help of strict inequality (2) we can prove Lemmas 4.2.2 and 4.2.3.

(b) Besides this strict inequality shows that if a and b belongs to sphere of ball $B[c,r]$ then z does not belong to this sphere, i.e., sphere does not contain straight lines therefore in previous definition we speak of M^*-convex round balls.

Counter-Examples. (i) We notice that well known metric space $\mathbb{R}$ with module metric and $\mathbb{R}^2$ with Euclidean metric is both strictly M^*-convex metric space and strictly M^*-convex metric space with M^*-convex round balls. But $\mathbb{R}^2$ with maximum metric is not strictly M^*-convex metric space. [Indeed, if we take $x = (0,0), y = (1,0), t = \frac{1}{2}$, the uniqueness is not satisfied because there exist two distinct points $z_1 = (\frac{1}{2}, \frac{1}{2}), z_1 = (\frac{1}{2}, 0)$ satisfy (1)].

(ii) Trivial example for strictly M^*-convex metric space that is not strictly M^*-convex metric space with M^*-convex round balls is space with one point x and $d(x,x) = 0$.

(iii) We notice that every convex subset of strictly convex Banach space is strictly M^*-convex metric space but no more strictly convex Banach space.

Strictly M^*-convex metric spaces with M^*-convex round balls inherent some good properties that we formulate as lemmas.

Lemma 4.2.2. ([Bul99], p. 9) *Let X be a strictly M^*-convex metric space with M^*-convex round balls and K be a M^*-convex and compact subset of X and $y \in X$. Then*

$$\exists \text{ a unique } y_0 \in K : d(y, y_0) = \inf\{d(x, y) : x \in K\}.$$

Proof. Define $T : K \to \mathbb{R}$ by the identity $T(x) := d(x, y), \forall\ x \in K$. Since K is a compact set, T attains its infimum on K and so $\exists\ y_0 \in K$:

$$T(y_0) = \inf\{T(x) : x \in K\} \text{ or } d(y, y_0) = \inf\{d(x, y) : x \in K\}.$$

The uniqueness we prove from contrary. We suppose that there exists another point $y_0' \in K$ such that $d(y; y_0') = \inf\{d(x, y) : x \in K\}$. The set K is M^*-convex, therefore for fixed $t \in]0, 1[$ exists such $y_0'' \in K$ that

$$d(y_0, y_0'') = td(y_0, y_0') \text{ and } d(y_0'', y_0') = (1 - t)d(y_0, y_0')$$

From condition of M^*-convex round balls follows that

$$d(y; y_0'') < \max\{d(y, y_0), d(y, y_0')\} = \inf\{d(x, y) : x \in K\}.$$

We have obtained that point y_0'' to be closer than points y_0 and y_0'. The contradiction completes the proof. $\square$

There is a certain course in fixed point theory formed by self-mappings of sets with "normal structure", a concept introduced by M. Brodskij and D. Milman [BM48] in 1948 (Section 2.1).

Definition: An M^*-convex set K in a metric space (X, d) is said to have **normal structur**e if for each bounded and M^*-convex subset $H \subseteq K$, that contains more than one point, there is some point $y \in H$ such that

$$\sup\{d(x, y) : x \in H\} < \delta(H).$$

Lemma 4.2.3. ([Bul99], p. 9) *Every M^*-convex and bounded set in strictly M^*-convex metric space X with M^*-convex round balls has normal structure.*

Proof. Suppose K is M^*-convex and bounded set in space$X$ that does not have normal structure. Then exists bounded and M^*-convex subset $H \subseteq K$ that contains more than one point and

$$\forall x \in H : \sup\{d(x, y) : y \in H\} = \delta(H).$$

We choose point $x_1 \in H$. Then $\exists\ x_2 \in H$ such that $d(x_1, x_2) = \delta(H)$. Since H is M^*-convex set then for fixed $t \in]0, 1[$ exists $z \in H$ such that

$$d(x_1, z) = td(x_1, x_2) \text{ and } d(z, x_2) = (1 - t)d(x_1, x_2).$$

Since $z \in H$ then $\exists\, x_3 \in H$ that $d(x_3, z) = \delta(H)$. But then by condition of M^*-convex round balls:

$$d(x_3, z) = \delta(H) < \max\{d(x_1, x_3), d(x_2, x_3)\} \leq \delta(H).$$

The contradiction completes the proof. $\square$

4.3 Fixed Points for Quasi-nonexpansive Maps in Strictly M^*-Convex Metric Spaces

In this section we present fixed point theorems for nonexpansive mapping and for commutative families of nonexpansive or quasi-nonexpansive or asymptotically nonexpansive mappings in strictly M^*-convex metric space with M^*-convex round balls as studied by Bula in [Bul99]. These results extend previous results of R. de Marr, F. E. Browder, W. A. Kirk, K. Goebel, W. G. Dotson,T. C. Lim and some others.

Recall that: A metric space X is said to be **strictly M^*-convex** if for each $x, y \in X$ and for each $t \in [0, 1]$ there exists unique $z \in X$ that satisfies:

$$d(x, z) = td(x, y) \text{ and } d(z, y) = (1 - t)d(x, y). \tag{1}$$

In addition to classic case we can prove that set of fixed points for nonexpansive self-mappings in M^*-convex closed subset of strictly M^*-convex metric space is M^*-convex and closed (I. Galina [Ga92]).

Lemma 4.3.1. ([Bul99], p. 10) *Let X be a strictly M^*-convex metric space and $K \subseteq X$, an M^*-convex and closed subset. If mapping $T : K \to K$ is a quasi-nonexpansive then the set of all fixed points of mapping T $Fix(T)$ is closed and M^*-convex.*

Proof. Since T is quasi-nonexpansive then $Fix(T) \neq \varnothing$ and T is continuous mapping in all fixed points. We assume, that $Fix(T)$ is not closed set. Then exists x that belongs to boundary of $Fix(T)$ and that does not belong to $Fix(T)$. Since K is closed set then $x \in K$.Since $x \notin Fix(T)$ then $T(x) \neq x$.We define $r := \frac{1}{3}d(T(x), x) > 0$.Then exists $y \in Fix(T)$ that $d(x, y) \leq r$. Since T is quasi-nonexpansive then $d(T(x), y) \leq d(x, y) \leq r$, and we have:

$$3r = d(T(x), x) \leq d(T(x), y) + d(y, x) \leq 2r.$$

This contradiction shows that assumption of $Fix(T)$ un-closedness is false.

Now we prove that $Fix(T)$ is M^*-convex set. We choose freely two points x and y $(x \neq y)$ in set $Fix(T)$. Let $t \in]0, 1[$. We find the corresponding $z \in K : d(x, z) =$

$td(x,y)$ and $d(z,y)=(1-t)d(x,y)$, which is unique by strictly M^*-convexity of X. We want to prove, that point z belongs to set $Fix(T)$.

Since T is quasi-nonexpansive then

$$d(T(z),x)<d(z,x) \text{ and } d(T(z),y)\leq d(z,y).$$

Therefore

$$\begin{aligned} d(x,y) &\leq d(x,T(z))+d(T(z),y)\leq d(z,x)+d(z,y) \\ &= td(x,y)+(1-t)d(x,y)=d(x,y). \end{aligned}$$

It follows that

$$\begin{aligned} d(x,T(z) &= d(z,x)=td(x,y), \\ d(T(z),y) &= d(z,y)=(1-t)d(x,y). \end{aligned}$$

By strictly M^*-convexity of X implies that $z=T(z)$ and $z\in Fix(T)$, i.e., $Fix(T)$ is M^*-convex set. $\square$

Lemma 4.3.2. ([Bul99], p. 10) *Let K be M^*-convex and closed subset of strictly M^*-convex metric space X. If mapping $T:K\to K$ is an asymptotically nonexpansive then the set of all fixed points of mapping T $Fix(T)$ is closed and M^*-convex.*

Proof. First**.** We prove that $Fix(T)$ is closed, it suffices to show that $\overline{Fix(T)}\subseteq Fix(T)$.Let $z\in\overline{Fix(T)}$, then there is a sequence $\{z_n\}\subseteq Fix(T)$ such that $\lim\limits_{n\to\infty} z_n=z$. Now, T, being asymptotically nonexpansive, is continuous. So

$$T(z)=T(\lim_{n\to\infty} z_n)=\lim_{n\to\infty} T(z_n)=\lim_{n\to\infty} z_n=z\ ().$$

So $z\in Fix(T)$. Hence $\overline{Fix(T)}\subseteq Fix(T)$ i.e., $Fix(T)$ is closed.

Second. We prove that $Fix(T)$ is M^*-convex set. We choose freely two points x and y $(x\neq y)$ in $Fix(T)$, then

$$T^i(x),T^i(y)\in Fix(T),\quad i=1,2,...$$

Let $t\in]0,1[$. We find the corresponding

$$z\in K: d(x,z)=td(x,y),\quad d(z,y)=(1-t)d(x,y). \text{ (since } K \text{ is } M^*\textit{-convex)} \tag{1}$$

Sine X is strictly M^*-convex then z is unique. We will have to prove that $z\in Fix(T)$ or $z=T(z)$.From definition of asymptotically nonexpansive mapping follows that:

$$d(T^i(z),x)=d(T^i(z),T^i(x))\leq k_i d(z,x)=tk_i d(x,y),\quad i=1,2,... \tag{2}$$

$$d(T^i(z),y)=d(T^i(z),T^i(y))\leq k_i d(z,y)=(1-t)k_i d(x,y),\quad i=1,2,.. \tag{3}$$

Inequality of triangle and (1) and (3) implies:

$$\begin{aligned} d(x,y) &\leq d(x,T^i(z)) + d(T^i(z),y) \\ &\leq tk_i d(x,y) + (1-t)k_i d(x,y) = d(x,y),\ i = 1,2,... \end{aligned}$$

Let i tend to infinity. Then $\lim\limits_{i\to\infty} k_i = 1$ and

$$d(x; \lim_{i\to\infty} T^i(z)) + d(\lim_{i\to\infty} T^i(z), y) = td(x,y) + (1-t)d(x,y).$$

From (2) and (3) follows that

$$\begin{aligned} d(x, \lim_{i\to\infty} T^i(z)) &= td(x,y), \\ d(\lim_{i\to\infty} T^i(z), y) &= (1-t)d(x,y). \end{aligned}$$

z is a unique point with property (1) therefore $\lim\limits_{i\to\infty} T^i(z) = z$. It follows that

$$z = \lim_{i\to\infty} T^i(z) = \lim_{i\to\infty} T^{i+1}(z) = T(\lim_{i\to\infty} T^i(z)) = T(z),$$

i.e., $z \in Fix(T)$ and $Fix(T)$ is M^*-convex set. □

Inspired from fixed point theorems where condition of normal structure is used (for example, R. de Marr [Marr63], W. A. Kirk [Kir65], W. Takahashi [Tak70] or M. R. Taskovic [Tas97]), Bula [Bul99] proved that:

Theorem 4.3.3. ([Bul99], p. 11) *Let X be strictly M^*-convex metric space with M^*-convex round balls. Let $K \subseteq X$ be M^*-convex and compact set. If $T : K \to K$ is nonexpansive mapping, then T has a fixed point in K.*

Proof. Let $D = \{B \subseteq K : B$ is nonempty, closed, M^*-convex and $T(B) \subseteq B\}$.Clearly $D \neq \varnothing$ (since $K \in D$).Let $\mathcal{B} = \{B_\alpha : \alpha \in I\}$ be a chain (totally ordered subset) in D. $A = \bigcap\limits_{\alpha\in I} B_a$ is closed, M^*-convex and $T(B) \subseteq B$. We need to show that A is nonempty. By definition of D, $B_\alpha \cap K \neq \varnothing$ for each α (since $B_\alpha \subseteq K$). Take the set $\{B_{\alpha_1} \cap K, B_{\alpha_2} \cap K\} \subseteq K$.

$$\cap_{i=1}^2 B_{\alpha_i} \cap K \neq \varnothing \text{ (since } B_{\alpha_1} \subseteq B_{\alpha_2} \text{ or } B_{\alpha_2} \subseteq B_{\alpha_1}).$$

By compactness of K,

$$\bigcap_{\alpha\in I} (B_a \cap K) \neq \varnothing \text{ or } \left(\bigcap_{\alpha\in I} B_a\right) \bigcap K \neq \varnothing.$$

So $\bigcap\limits_{\alpha\in I} B_a \neq \varnothing$.Then $\bigcap\limits_{\alpha\in I} B_a$ is a lower bound of $\mathcal{B}$. Hence by Zorn's Lemma D has a minimal $K_0 \subseteq K$.

We show that K_0 consists of a single point. We assume that $\delta(K_0) > 0$. Since K_0 is M^*-convex set then by Lemma 4.2.3 K_0 has normal structure, i.e.,

$$\exists x \in K_0 : \sup\{d(x,y) : y \in K_0\} = r < \delta(K_0).$$

We denote M^*-convex closed hull of set $T(K_0)$ with $\overline{co}T(K_0) = K_1$. Since $T(K_0) \subseteq K_0$ then

$$\begin{aligned} K_1 &= \overline{co}T(K_0) \subseteq K_0 \text{ and} \\ T(K_1) &\subseteq T(K_0) \subseteq \overline{co}T(K_0) = K_1. \end{aligned}$$

The minimality of K_0 implies $K_1 = K_0$.

We define set

$$C := (\cap_{y \in K_0} B[y, r]) \cap K_0.$$

That is nonempty since $x \in C$, that is M^*-convex (by Lemma 4.2.1 in Section 4.2, balls are M^*-convex sets) and closed set as intersection of M^*-convex and closed sets.

We define set

$$C_1 := (\cap_{y \in T(K_0)} B[y; r]) \cap K_0.$$

Since $T(K_0) \subseteq K_0$, then $C \subseteq C_1$. [Indeed, let $z \in C$,then $z \in K_0$ and

$$\begin{aligned} &\Longrightarrow z \in \cap_{y \in K_0} B[y, r] \\ &\Longrightarrow z \in \cap_{y \in K_0} B[y, r] \\ &\Longrightarrow z \in B[y, r] \forall y \in K_0 \\ &\Longrightarrow z \in B[y, r] \forall y \in T(K_0) \quad (\text{since } T(K_0) \subseteq K_0) \\ &\Longrightarrow z \in \cap_{y \in T(K_0)} B[y, r]. \end{aligned}$$

But $z \in K_0$. Therefore $C \subseteq C_1$.]

Since $T(K_0) \subseteq B[z; r]$ and $K_0 = K_1 = \overline{co}T(K_0) \subseteq B[z; r]$ then $C_1 \subseteq C$. [Indeed, let $z \in C_1$,then $z \in K_0$ and

$$\begin{aligned} &\Longrightarrow z \in \cap_{y \in T(K_0)} B[y, r] \\ &\Longrightarrow z \in \cap_{y \in T(K_0)} B[y, r] \\ &\Longrightarrow z \in B[y, r] \ \forall \ y \in T(K_0) \\ &\Longrightarrow d(y, z) \leq r \ \forall \ y \in T(K_0). \end{aligned}$$

Now $K_0 = K_1 = \overline{co}T(K_0) \subseteq B[z; r]$ (because $B[z, r]$ is closed and M^*-convex set). Thus $K_0 \subseteq B[z; r]$.It follows that $d(y, z) \leq r \ \forall \ y \in K_0$,

$$\begin{aligned} &\Longrightarrow z \in B[y, r] \forall y \in K_0 \\ &\Longrightarrow z \in \cap_{y \in K_0} B[y, r] \end{aligned}$$

But $z \in K_0$. Therefore $C_1 \subseteq C$.] Hence $C = C_1$.

We choose $z \in C$ and $y \in T(K_0)$. Then exists $x \in K_0$ such that $y = T(x)$. Thereby:

$$d(T(z), y) = d(T(z), T(x)) \leq d(z; x) \leq r;$$

i.e.,$T(z) \in C_1$. Since $C = C_1$ then $T(z) \in C$ or $T(C) \subseteq C$. The minimality of K_0 implies $C = K_0$.But

$$\delta(C) \leq r \leq \delta(K_0).$$

From obtained contradiction, we conclude that $\delta(K_0) = 0$ and $K_0 = \{x^*\}$ and therefore$T(x^*) = x^*$. □

We generalize Theorem 4.3.3 for commutative family of nonexpansive mappings.

Definition: A family of mappings F is **commutative** if for all $x \in K$, where K is an arbitrary set, condition

$$f(g(x)) = g(f(x))$$

holds for all $f; g \in F$.

These result generalize fixed point theorems for commutative family of nonexpansive mappings of R. de Marr [Marr63], F. E. Browder [Brow65] and T. C. Lim [Lim74].

Theorem 4.3.4. ([Bul99], p. 13) *Let X be a strictly M^*-convex metric space with M^*-convex round balls. Let $K \subseteq X$ is M^*-convex and compact set. If $F = \{f : f : K \to K\}$ is commutative family of nonexpansive mappings then exists a common fixed point for family F, i.e.,*

$$\exists x^* \in K \;\; \forall f \in F : \quad f(x^*) = x^*.$$

Proof. From Theorem 4.3.3, it is known that $Fix(f) \neq \varnothing$, $\forall\ f \in F$. Since X is strictly M^*-convex metric space, by Lemma 4.3.1, $Fix(f)$ is M^*-convex and closed sets for every $f \in F$ (since $Fix(f) \neq \varnothing$, every nonexpansive mapping is quasi-nonexpansive).

Let us inductively prove that $\cap_{i=1}^{n} Fix(f_i) \neq \varnothing$ for every $n \in N$. For $n = 1$ the statement is true from the Theorem 4.3.3 Assuming that $\cap_{i=1}^{k} Fix(f_i) \neq \varnothing$, let us prove that $\cap_{i=1}^{k+1} Fix(f_i) \neq \varnothing$.Since by assumption

$$f_{k+1}(x) = f_{k+1}(f_i(x)) = f_i(f_{k+1}(x)), \quad i = 1, 2, ..., k;$$

it follows that

$$\begin{aligned} f_{k+1}(x) \;\; &\in \;\; \cap_{i=1}^{k} Fix(f_i); \\ \text{and hence } f_{k+1} \;\; &: \;\; \cap_{i=1}^{k} Fix(f_i) \; \to \cap_{i=1}^{k} Fix(f_i). \end{aligned}$$

Let us prove that the mapping f_{k+1} has a fixed point in the set $\cap_{i=1}^{k} Fix(f_i)$.The sets $Fix(f_i)$, $i = 1, 2, ...k$, are nonempty, closed and M^*-convex, therefore $\cap_{i=1}^{k} Fix(f_i)$ is closed and M^*-convex as intersection of closed and M^*-convex sets; that is compact set as closed subset of compact set K. By Theorem 4.3.3 for nonexpansive mapping , there exists

$$f_{k+1} : \cap_{i=1}^{k} Fix(f_i) \; \to \cap_{i=1}^{k} Fix(f_i)$$

has fixed point in set $\cap_{i=1}^{k} Fix(f_i)$, therefore

$$\cap_{i=1}^{k+1} Fix(f_i) \neq \varnothing.$$

Since set K is compact, $\cap_{f\in F} Fix(f)$ is nonempty set also for infinite family of mappings. □

Similar theorems we can to prove for commutative family of quasi-nonexpansive and asymptotically nonexpansive mappings.

Theorem 4.3.5. ([Bul99], p. 13) *Let X be a strictly M^*-convex metric space with M^*-convex round balls. Let $K \subseteq X$ is M^*-convex and compact set. If $F = \{f : f : K \to K\}$ is commutative family of quasi-nonexpansive mappings then exists a common fixed point for family F.*

Proof. Idea of proof is similar as in Theorem 4.3.4 The differences are that for $n = 1$ the statement is true by definition of quasi-nonexpansive mapping and the proof that mapping f_{k+1} has a fixed point in the set $\cap_{i=1}^{k} Fix(f_i)$.

Let

$$f_{k+1} : \cap_{i=1}^{k} Fix(f_i) \ \to \cap_{i=1}^{k} Fix(f_i).$$

Let $\cap_{i=1}^{k} Fix\ f_i$ be M^*-convex and compact set (this follows from Lemma 4.3.1). We fix $z \in Fix(f_{k+1}) \neq \varnothing$. By Lemma 4.2.2 in Section 4.2 there exists unique element

$$z_0 \in \cap_{i=1}^{k} Fix(f_i) \ \text{ such that } d(z, z_0) = \inf\{d(z, y) \ : \ y \in \cap_{i=1}^{k} Fix(f_i)\}.$$

Then from definition of quasi-nonexpansive mapping we get:

$$\begin{aligned} d(z, z_0) &= \inf\{d(z, y) \ : \ y \in \cap_{i=1}^{k} Fix(f_i)\} \\ &\leq d(z, f_{k+1}(z_0)) = d(f_{k+1}(z), f_{k+1}(z_0)) \leq d(z, z_0). \end{aligned}$$

From uniqueness of z_0 follows that $f_{k+1}(z_0) = z_0$, therefore

$$z_0 \in \cap_{i=1}^{k+1} Fix(f_i) \neq \varnothing. \ \square$$

Theorem 4.3.6. ([Bul99], p. 14) *Let X be a strictly M^*-convex metric space with M^*-convex round balls. Let $K \subseteq X$ is M^*-convex and compact set. If $\mathcal{F} = \{f : f : K \to K\}$ is commutative family of asymptotically nonexpansive mappings and*

$$\forall f \in \mathcal{F} : Fix(f\ ) \neq \varnothing$$

then exists a common fixed point for family $\mathcal{F}$.

Proof. Proof is similar to previous theorems. The differences are following. The set $\cap_{i=1}^{k} Fix f_i$ is M^*-convex and closed by Lemma 4.3.2, since K is M^*-convex and compact set then $\cap_{i=1}^{k} Fix(f_i)$ is M^*-convex and compact set. We prove that mapping

$$f_{k+1} : \cap_{i=1}^{k} Fix(f_i) \to \cap_{i=1}^{k} Fix(f_i)$$

has a fixed point in set $\cap_{i=1}^{k} Fix(f_i)$. We fix $z \in Fix(f_{k+1}) \neq \varnothing$. By lemma 4.2.2 there exists unique element $z_0 \in \cap_{i=1}^{k} Fix(f_i)$ such that

$$d(z, z_0) = \inf\{d(z, y) \ : \ y \in \cap_{i=1}^{k} Fix(f_i)\}.$$

Then

$$\begin{aligned} d(z, z_0) &= \inf\{d(z, y) \ : \ y \in \cap_{i=1}^{k} Fix(f_i)\} \leq d(z, f_{k+1}^{i}(z_0)) \\ &= d(f_{k+1}^{i}(z), f_{k+1}^{i}(z_0)) \leq k_i d(z, z_0), \quad i = 1, 2, ... \end{aligned}$$

Let $i \to \infty$.Then

$$\lim_{i\to\infty} k_i = 1 \text{ and } d(z, z_0) = d(z, \lim_{i\to\infty} f_{k+1}^{i}(z_0)).$$

From uniqueness of z_0 follows that

$$z_0 = \lim_{i\to\infty} f_{k+1}^{i}(z_0)$$

Since, by continuity of asymptotically nonexpansive mapping,

$$z_0 = \lim_{i\to\infty} f_{k+1}^{i}(z_0) = \lim_{i\to\infty} f_{k+1}^{i+1}(z_0) = f_{k+1}(\lim_{i\to\infty} f_{k+1}^{i}(z_0)) = f_{k+1}(z_0)$$

then

$$z_0 \in \cap_{i=1}^{k+1} Fix(f_i) \neq \varnothing. \ \square$$

4.4 Best Approximation in Strict M-Convex metric spaces

In this section, we consider some results of Narang [Nar81]. The notion of strict M-convexity in metric spaces was introduced in [ANT74] and certain existence and uniqueness theorems on best approximation in such a space were proved in [ANT74] and [ANT78]. This section deal with a stronger version of the notion of strict M-convexity and characterize such metric spaces, as introduced in [Nar81]. Further, we consider the unicity theorem of best approximation 'Every $M-$convex proximinal set in a strictly M-convex metric space is Chebyshev'.

Definition: Let (X, d) be a metric space and $x, y, z \in X$. We say that the point z is **between** x and y (writing xzy) if

$$d(x, z) + d(z, y) = d(x, y).$$

Thereby, a metric space (X, d) is said to be M-**convex** [Rolf67] if for every two points x and $y \in X$, there exists $z \in X$ such that $x \neq y \neq z$ and xzy.

For any two points x, y of X, the set of all points which lie between x and y, is called the segment $[x, y]_M$.i.e,

$$[x, y]_M = \{z \in X : d(x, z) + d(z, y) = d(x, y)\},$$

Theorem 4.4.1. ([Cha80], p. 43) *Every normed vector space is an M-convex metric space.*

Proof. Let $X = (X, \|.\|)$ be a normed vector space and let $x, y \in X, x \neq y$. Since X is linear space, $\exists\ z = \frac{x+y}{2} \in X$. Now

$$\begin{aligned} d(x, z) + d(z, y) &= \|x - z\| + \|z - y\| \\ &= \left\|x - \frac{x+y}{2}\right\| + \left\|\frac{x+y}{2} - y\right\| \\ &= \left\|\frac{2x - x - y}{2}\right\| + \left\|\frac{x + y - 2y}{2}\right\| \\ &= \left\|\frac{x-y}{2}\right\| + \left\|\frac{x-y}{2}\right\| \\ &= \frac{1}{2}\|x - y\| + \frac{1}{2}\|x - y\| \\ &= \|x - y\| = d(x, y). \end{aligned}$$

Thus, X is an M-convex metric space.

We now give two examples to illustrate the fact that not every metric space is M-convex and also the fact not every M-convex metric space is a normed vector space.

Example. Let K be a non-convex subset of $\mathbb{R}^2$equipped with the usual Euclidean metric. Then it is easy to see that K is a metric space which is not M-convex.

Solution. Suppose

$$K = \{(1, y) : 0 \leq y \leq 2\} \cup \{(x, 2) : 1 \leq x \leq 3\} \cup \{(3, y) : 0 \leq y \leq 2\}.$$

Take $x = (1, 0), y = (3, 0) \in K$. Only $z = (2, 0)$ satisfies $d(x, y) = d(x, z) + d(z, y)$.But $z \notin K$. So K is not M-convex.

Example. The ball

$$B = B[0, 1] = \{x \in \mathbb{R}^2 : d(x, 0) \leq 1\} = \{x \in \mathbb{R}^2 : ||x|| \leq 1\}$$

in $\mathbb{R}^2$ with the usual Euclidean metric is an M-convex metric space which is not a normed vector space.

Solution. To show that B is M-convex, let $x = (x_1, x_2), y = (y_1, y_2) \in B$. Take $z = \frac{x+y}{2} = (\frac{x_1+y_1}{2}, \frac{x_2+y_2}{2})$. Then $z \in B$, since

$$||z|| = ||\frac{x + y}{2}|| \leq \frac{1}{2}(||x|| + ||y||) \leq \frac{1}{2}(1 + 1) = 1.$$

Further,

$$\begin{aligned}
& d(x,z)+d(z,y)\\
= & \sqrt{\left(x_1-\frac{x_1+y_1}{2}\right)^2+\left(x_2-\frac{x_2+y_2}{2}\right)^2}\\
& +\sqrt{\left(\frac{x_1+y_1}{2}-y_1\right)^2+\left(\frac{x_2+y_2}{2}-y_2\right)^2}\\
= & 2\sqrt{\frac{(x_1-y_1)^2}{4}+\frac{(x_2-y_2)^2}{4}}\\
= & \sqrt{(x_1-y_1)^2+(x_2-y_2)^2}=d(x,y).
\end{aligned}$$

But B is not a normed vector space since B is not a vector space. □

Definition: [Rolf67] A metric space (X,d) is called **strongly M-convex** if for any $x,y\in X$, with $x\neq y$, there exists a unique $z\in X$ such that

$$d(x,y)=d(x,z)+d(y,z).$$

Definition: A metric space (X,d) is said to be **strictly M-convex** if for every $x\neq y$ in X, and $r>0$,

$$\begin{aligned}
d(x,y) &= d(x,z)+d(z,y) \text{ (i.e. strongly } M\text{-convex)}\\
\text{and } d(p,x) &\leq r, d(p,y)\leq r\Rightarrow d(p,z)<r.
\end{aligned}$$

where p is arbitrary but fixed point of X and z is any point in the open segment $]x,y[$ (i.e. $z\neq x, z\neq y$).

Therefore, in a strictly M-convex metric space if x and y are any two points on the boundary of a sphere, then $]x,y[$ lies strictly inside the sphere, i.e. $d(z,x)<r$, $d(z,y)<r$.

Note. If X is a normed vector spaces, taking $p=0\in X$ and $z=\frac{x+y}{2}$in the above definition, we obtain

$$||x||\leq r, ||y||\leq r\Rightarrow ||\frac{x+y}{2}||<r.$$

Hence X becomes a strict convex normed space in the usual sense (Section 1.5).

Definition: [ANT74] A subset K of a metric space (X,d) is said to be M-**convex** if for every $x,y\in K$, any point between x and y is also in K i.e. for each x,y in K, the segment $[x,y]_M$ lies in K.

We have the following unicity theorem of best approximation.

Theorem 4.4.2. ([Nar81], p. 88; [ANT74], p. 95) *In a strictly M-convex metric space X, every proximinal convex set is Chebyshev.*

Proof. Suppose that K is a convex proximinal subset of X. Let $x \in X$. We want to show K is Chebyshev (i.e., $P_K(x)$ is a singleton set). Since K is proximinal, $P_K(x) \neq \varnothing$. Let $z_1, z_2 \in P_K(x)$, and $d(x, K) = r$. Then

$$d(z_1, x) = d(x, K) = r, \tag{1}$$

and

$$d(z_2, x) = d(x, K) = r. \tag{2}$$

Let z^* is midpoint of z_1 and z_2. Since K is convex and $z_1, z_2 \in K$, $z^* \in K$. By strict M-convexity of X,

$$d(z^*, x) < r = d(x, K) \text{ unless } z_1 = z_2. \tag{3}$$

If $z_1 \neq z_2$,(3) will be contradicted (1) and (2) (because, by definition of $d(x, K)$, $d(x, K) \leq d(x, z)$ for all $z \in K$). Therefore $z_1 = z_2$. Hence K is Chebyshev. □

Lemma 4.4.3. ([Nar81], p. 88) *For any two points x, y in a strongly M-convex metric space (X, d), the segment $[x, y]_M$ is convex.*

Proof. Since X is strongly M-convex for each pair x and y there exist a unique z between them (i.e. $z \in [x, y]_M$). Let $[x, y]_M = K$ and $z, y \in K$. To show that $[z, y]_M \in K$, let $z' \in [z, y]_M$.Then

$$\begin{aligned} d(x, y) &= d(x, z) + d(z, y) = d(x, z) + d(z, z') + d(z', y) \\ &\geq d(x, z') + d(z', y) \geq d(x, y). \end{aligned}$$

Hence

$$d(x, y) = d(x, z') + d(z', y).$$

So $z' \in [x, y]_M$.Therefore $[z, y]_M \subseteq K = [x, y]_M$. □

In order to show that the converse of the above theorem also holds, we establish a lemma.

Lemma 4.4.4. ([Nar81], p. 88) *For any two points x, y in a strongly M-convex metric space (X, d), consider the function $\phi_x : [x, y]_M \to [0, d(x, y)] \subseteq \mathbb{R}$* defined by

$$\phi_x(z) = d(x, z), \;\; z \in [x, y]_M.$$

Then ϕ_x is an isometry.

Proof. We can assume $x \neq y$. Let $z \in [x, y]_M$ and $z' \in [z, y]_M$. Then

$$\begin{aligned} d(x, y) &= d(x, z) + d(z, y) = \\ &= d(x, z) + d(z, z') + d(z', y) \geq d(x, z') + d(z', y) \geq d(x, y). \end{aligned}$$

Hence

$$d(x, y) = d(x, z') + d(z', y) \tag{1}$$

and

$$\phi_x(z') = d(x, z') = d(x, z) + d(z, z') = \phi_x(z) + d(z, z')$$

implying

$$\left|\phi_x(z') - \phi_x(z)\right| = d(z, z'). \tag{2}$$

The equality (1) shows that $z' \in [x, y]_M$ and the equality (2) shows that ϕ_x is an isometry. □

Corollary 4.4.5. ([Nar81], p. 88) *For any two points x, y in a strongly M-convex metric space (X, d) the segment $[x, y]_M$ is a compact set.*

Proof. The map $\phi_x : [x, y]_M \to [0, d(x, y)]$ is an isometry and so its inverse ${\phi_x}^{-1} : [0, d(x, y)] \to [x, y]_M$ is continuous. Since $[0, d(x, y)]$ is compact, $[x, y]_M$ is also compact. □

Notation. Let S be a subset of a metric space (X, d). For any $z \in S$, let

$$Q_S(z) = \{x \in X : d(x, z) = d(x, S)\},$$

i.e $Q_S(z)$ is the set of all those points of X having z as a nearest point in S.

The following theorem shows that the converse of the unicity Theorem (Theorem 4.4.2) is also true. This result is also an analogue of ([Khal88, Theorem 2.6).

Theorem 4.4.6. ([Nar81], p. 89) *Let (X, d) be a strongly M-convex metric space. Then the following statements are equivalent:*

(i) *X is strictly M-convex.*

(ii) *For each convex set S and distinct points x and y of S, $Q_S(x) \cap Q_S(y) = \varnothing$.*

(iii) *Every convex proximinal set S is Chebyshev.*

Proof. (i)$\Longrightarrow$(ii) Let, if possible, $Q_S(x) \cap Q_S(y) \neq \varnothing$ and let $z \in Q_S(x) \cap Q_S(y)$.This implies,

$$d(z, x) = d(z, y) = d(z, S).$$

Now $x, y \in X$ and X is a strongly M-convex space, therefore there exists a unique $q \in X$ such that xqy. Then $q \in [x, y]_M$, and since S is a convex set, $[x, y]_M \subseteq S$. Therefore $q \in S$.

Strict M-convexity of the space implies $d(z, q) < d(z, S)$, which is a contradiction.

(ii)$\Longrightarrow$(iii) Let a convex set S be proximinal. Let $p \in X$. Since S is proximinal, $P_S(p) \neq \varnothing$. Then there exists $x \in S$ such that

$$d(x, p) = d(p, S),$$

i.e.$p \in Q_S(x)$.

Let if possible, $y \neq x$ be also nearest to p, then

$$d(y, p) = d(p, S).$$

Then $p \in Q_S(y)$. Thus $p \in Q_S(x) \cap Q_S(y)$, $x \neq y$, which is a contradiction.

(iii)$\Longrightarrow$(i) Let $x \neq y$, p be points of (X, d) with $d(x, p) = d(y, p) = r$ (say). Define $f : I = [0, d(x, y)] \to \mathbb{R}$ by

$$f(t) = d(p, \phi_x^{-1}(t)).$$

Then f is continuous. Moreover, since $[x, y]_M$ is a compact convex subset, it is proximinal. The hypothesis (iii) implies that there exists no subinterval $[t_1, t_2] \subseteq I$, $t_1 < t_2$, such that

$$f(t_1) = f(t_2) = \min\{f(t) : t_1 \leq t \leq t_2\}.$$

We affirm that all interior points $t \in]0, d(x, y)[$ satisfy

$$f(t) < \max f = f(0) = f(d(x, y)). \tag{3}$$

Let, if possible, $f\ (t_0) \geq \max f$ for some interior point. Set

$$\begin{aligned} m' &= \min\{f(t) : t \leq t_0\}, \\ m'' &= \min\{f(t) : t \geq t_0\}. \end{aligned}$$

Suppose $m' \geq m''$. Define

$$\begin{aligned} t_0' &= \inf\{t : t \leq t_0, \min\{f(t_1) : t \leq t_1 \leq t_0\} \geq m'\}, \\ t_0'' &= \sup\{t : t \geq t_0, \min\{f(t_2) : t_0 \leq t_2 \leq t\} \geq m''\}; \end{aligned}$$

Since f is continuous it follows that

$$f(t_0') = f(t_0'') = \min\{f(t) : t_0' \leq t \leq t_0''\}.$$

If $t_0' < t_0''$ then $[t_0', t_0'']$ is the subinterval leading to a contradiction; if $t_0' = t_0''$ then $I = [0, d(x, y)]$ is the subinterval leading to a contradiction. If $m' \leq m''$, a contradiction can be reached by a similar argument.

Since ϕ_x is an isometry, (3) implies $d(z, p) < r$ for any point z in the open segment $]x, y[$. Hence the space is strictly M-convex. $\square$

Chapter 5

Hyperconvex, CAT(0), $\mathbb{R}$-trees and Hyperbolic spaces

In this chapter we deal with Hyperconvex spaces and its various properties. Also we include its applications in fixed point theory. Furthermore, we present CAT(0) spaces and its applications in fixed point sets. Also, we shed light on $\mathbb{R}$-trees and the connection between them and CAT(0) spaces. Finally, we make a study of Hyeperbolic spaces and fixed point results.

5.1 Hyperconvex Spaces and Nonexpansive Extensions

The notion of hyperconvex spaces has been introduced by N. Aronszahn and P. Panitchpakdi [AroPan56] in 1956. In this section we study hyperconvex spaces and their relationships to nonexpansive extension of mappings. Moreover, we present some properties of hyperconvexity. These include completeness. These results are due to Baillon in [Bail88].

Definition: A metric space M is called **metrically convex (in short, $M^{\oplus}$-convex**) if for any two distinct points, x and y, and for any decomposition $d(x, y) = r + s$, $r > 0, s > 0$ there exists a point z in M with $d(x, z) = r$ and $d(y, z) = s$.

Definition: A metric space (M, d) is said to be **hyperconvex** if

$$\bigcap_{\alpha \in A} B[x_\alpha, r_\alpha] \neq \varnothing$$

for any collection $\{B[x_\alpha, r_\alpha] : \alpha \in A\}$ of closed balls in M for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$.

Remark. Hyperconvexity $\Rightarrow$ $M^{\oplus}$-convexity.

Proof. Suppose (M, d) is hyperconvex To show that (M, d) is $M^{\oplus}$-convex, consider the decomposition $d(x, y) = r + s$, $r > 0, s > 0$. Applying hyperconvexity to $d(x, y) =$

$r+s$, choose a $z \in B[x,r] \cap B[y,s]$. Then it follows that

$$d(x,z) \leq r, d(y,z) \leq s. \qquad (1)$$

On the other hand,

$$r+s = d(x,y) \leq d(x,z) + d(y,z). \qquad (2)$$

Hence, by (1) and (2),

$$\begin{aligned} d(x,z) &\geq r+s-d(y,z) \text{ by } (2) \\ &\geq r+0 = r \quad \text{by } (1). \end{aligned}$$

Similarly $d(y,z) \geq s$. Thus $d(x,z) = r$ and $d(y,z) = s$. □

Definition: A metric space is said to have the **binary intersection property** if for any set of closed balls $B[x_i, r_i]$, $i \in I$, such that every two of these balls intersect, then all the balls intersect.

We motivate the study of hyperconvex spaces with a simple observation about the real line $\mathbb{R}$.

Theorem 5.1.1. ([KK01], p. 71) *Let $\{I_\alpha\}_{\alpha \in A}$ be a family of bounded closed intervals of $\mathbb{R}$ each two of which intersect. Then*

$$\bigcap_{\alpha \in A} I_\alpha \neq \varnothing.$$

Proof. Suppose the intersection is empty. Then by compactness there exist

$$\{I_1, ..., I_{N+1}\} \subseteq \{I_\alpha\}$$

such that

$$I = \bigcap_{i=1}^{N} I_i \neq \varnothing \text{ while } \bigcap_{i=1}^{N+1} I_i = \varnothing.$$

Thus I and I_{N+1} are disjoint closed intervals in $\mathbb{R}$. Select any point x that lies strictly between them. (This is possible because the complement of $I \cup I_{N+1}$is an open set.) Then by the fact that any two members of the original family intersect,

$$x \in I_i \cap I_{N+1}, i = 1, ..., N.$$

This implies

$$x \in \bigcap_{i=1}^{N+1} I_i,$$

which is a contradiction. □

Remarks. (1) ([EK01], p. 393) Note that an closed bounded interval $I \subseteq \mathbb{R}$ may also be seen on the real line as a closed ball. [Indeed, the interval $I = [a, b]$ is also the closed ball centered at $x = (a + b)/2$ with radius $r = (b - a)/2$, i.e.

$$[a, b] = B\left(\frac{a+b}{2}, \frac{b-a}{2}\right) = B(x, r).$$

(2) If $z \in B[x_\alpha, r_\alpha] \cap B[x_\beta, r_\beta]$, then

$$d(x_\alpha, x_\beta) \leq d(x_\alpha, z) + d(z, x_\beta) \leq r_\alpha + r_\beta.$$

(3) In real line $\mathbb{R}$, if x_α, x_β and r_α, r_β are centers and radii of any two closed bounded intervals, respectively such that $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$. Then the two intervals must intersect, i. e. $[x_\alpha - r_\alpha, x_\alpha + r_\alpha] \cap [x_\beta - r_\beta, x_\beta + r_\beta] \neq \varnothing$[(since $\mathbb{R}$ is $M^\oplus$-convex).

Note. By Theorem 5.1.1 and Remarks (1) & (3), the real line $\mathbb{R}$ is hyperconvex space.

The next theorem confirms that hyperconvexity is equivalent to the binary intersection property and the metric convexity property.

Theorem 5.1.2. ([KK01], p. 79) *Let (M, d) be a complete metric space. Then the following are equivalent:*

(1) *M is hyperconvex.*

(2) *M is $M^\oplus$-convex and has the binary ball intersection property.*

Proof. (1)$\Longrightarrow$(2) This is already proved above in a Remark. It is obvious that hyperconvex spaces have binary ball intersection property.

(2)$\Longrightarrow$(1) Next suppose M is a complete metric space which has the binary ball intersection property, and suppose M is $M^\oplus$-convex. If $\{B[x_i, r_i]\}_{i \in I}$ is a family of closed balls in M for which

$$d(x_i, x_j) \leq r_i + r_j,$$

then by Theorem 2.16 of ([KK01], p. 35), there is a metric segment joining x_i and x_j and clearly some point of this segment must lie in $B[x_i, r_i] \cap B[x_j, r_j]$. Therefore, $\bigcap_{i \in I} B[x_i, r_i] \neq \varnothing$ by the binary ball intersection property. $\square$

Remarks. (1) Completeness is not actually necessary for the above, because that hyperconvex spaces and any space having binary ball intersection property are complete.(We will see that later on)

(2) In linear space, (2) reduces to the binary-intersection property on balls, because metric convexity holds.

For a subset A of a metric space (M, d), set:

$$\begin{aligned} r_x(A) &= \sup\{d(x,y) : y \in A\} \text{ for } x \in M, \\ r_M(A) &= \inf\{r_x(A) : x \in M\}, \\ r(A) &= \inf\{r_x(A) : x \in A\}, \\ \delta(A) &= \sup\{r_x(A) : x \in A\}, \\ C(A) &= \{x \in A : r_x(A) = r(A)\}, \\ cov(A) &= \cap\{B : B \supset A \text{ and } B \text{ a closed ball}\}. \end{aligned}$$

$r_M(A)$ is called the radius of A (relative to M), $\delta(A)$ is the diameter of A and $C(A)$ is called the Chebyshev center of A.

Properties of Hyperconvex Spaces

Theorem 5.1.3. ([Bail88], p. 12; [KK01], p. 80) *Any hyperconvex metric space is complete.*

Proof. Let $\{x_n\}$ be a Cauchy sequence in hyperconvex metric space and $\rho_n = \sup_{m \geq n} d(x_m, x_n)$. Then for $m \geq n$

$$d(x_n, x_m) \leq \rho_n \leq \rho_n + \rho_m.$$

Then $B[x_n, \rho_n] \cap B[x_m, \rho_m] \neq \varnothing$. By using the hyperconvexity property, $\exists\ z \in \bigcap_{n=1}^{\infty} B[x_n, \rho_n]$. Clearly $\lim_{n\to\infty} \rho_n = 0$.[Indeed, since $n \to \infty$, $m \to \infty$. Then

$$\lim_{n\to\infty} \rho_n = \lim_{n\to\infty} \sup_{m\geq n} d(x_m, x_n) \leq \sup_{m\geq n} \lim_{n\to\infty} d(x_m, x_n) = 0 \text{ (since } \{x_n\} \text{ is Cauchy).}$$

Thus $\lim_{n\to\infty} x_n = z$ [since $z \in B[x_n, \rho_n]$ for all n, $d(x_n, z) \leq \rho_n$ for all n.Then $\lim_{n\to\infty} d(x_n, z) = 0$.] □

Remark. Any space with binary ball intersection property is complete.

Theorem 5.1.4. ([Bail88], p. 12; [KK01], p. 80) *If A is a bounded subset of a hyperconvex space M, then we have the following relations:*

(1) $cov(A) = \cap\{B[x, r_x(A)] : x \in M\}$.

(2) $r_x(cov(A)) = r_x(A)$, for any $x \in M$.

(3) $r_M(cov(A)) = r_M(A)$.

(4) $r_M(A) = \frac{1}{2}\delta(A)$

(5) $\delta(cov(A)) = \delta(A)$.

Proof. (1) Since $B[x, r_x(A)]$ contains A for each $x \in M$ and $cov(A)$ is the smallest closed ball containing A it must be the case that

$$cov(A) \subseteq \{B[x, r_x(A)] : x \in M\}.$$

On the other hand, if $A \subseteq B[x,r]$, then $d(x,y) \leq r \ \forall \ y \in A$ so $r_x(A) \leq r$.Thus $B[x, r_x(A)] \subseteq B[x,r]$.

Hence

$$\cap\{B[z, r_z(A)] : z \in M\} \subseteq B[x,r].$$

Since $B[x,r]$ is any closed ball containing A and $cov(A)$ is a closed ball.This clearly implies

$$cov(A) = \cap\{B[x, r_x(A)] : x \in M\}.$$

(2) By 1, $r_x(cov(A)) = \sup\{d(x,y) : y \in \cap_{z\in M} B[z, r_z(A)]\}$. In particular, $y \in cov(A)$ implies $y \in B[x, r_x(A)]$, for any $x \in M$. Hence

$$d(x,y) \leq r_x(A).$$

This proves $r_x(cov(A)) \leq r_x(A)$. The reverse inequality is obvious Since $A \subseteq cov(A)$.

(3) This is immediate from the definition of r. [Indeed,

$$\begin{aligned} r_M(cov(A)) &= \inf\{r_x(cov(A)) : x \in M\} \\ \text{and } r_M(A) &= \inf\{r_x(A) : x \in M\}. \end{aligned}$$

By 2, $r_x(cov(A)) = r_x(A)$.This proves $r_M(cov(A)) = r_M(A)$.]

(4) Let $\delta = \delta(A)$ and consider the family $\{B[a, \frac{\delta}{2}] : a \in A\}$. If $a, b \in A$,then $d(a,b) \leq \delta = \frac{\delta}{2} + \frac{\delta}{2}$ so by hyperconvexity

$$\bigcap_{a\in A} B[a, \frac{\delta}{2}] \neq \varnothing.$$

If x is any point in this intersection then $d(x,a) \leq \frac{\delta}{2}$ so $r_x(A) \leq \frac{\delta}{2}$.On the other hand

$$d(a,b) \leq d(a,z) + d(z,b)$$

for any $a, b \in A$ and $z \in M$. By taking sup on A of both sides, we get

$$\delta \leq r_z(A) + r_z(A)$$

so $\delta \leq 2r_z(A)$ from which $\delta \leq 2r_M(A)$. Therefore,

$$\delta \leq 2r_M(A) \leq 2r_x(A) \leq \delta,$$

proving $r_M(A) = \frac{1}{2}\delta$.

(5) Using (3) and (4), $\delta(A) = 2r_M(A) = 2r_M(cov(A)) = \delta(cov(A))$. $\Box$

Definition: ([KK01], p.78) A metric space M is said to be **injective** if it has the following extension property:

Whenever Y is a subspace of a metric space X and $f : Y \to M$ is nonexpansive, then f has a nonexpansive extension $\widetilde{f} : X \to M$.

The relations between hyperconvex spaces and nonexpansive mappings are given by the following theorem:

Theorem 5.1.5. ([AroPan56], p. 416; [KK01], p. 81) *A metric space (M, d) is injective iff it is hyperconvex.*

Proof. ($\Longrightarrow$) Assume (M, d) is injective. In order to prove that M is hyperconvex, we need only to show that M is $M^{\oplus}$-convex and has the binary intersection property. Let $x, y \in M$ with $x \neq y$. Let Y be the metric subspace of M consisting of the points $\{x, y\}$ and let (X, ρ) be the metric space consisting of the points $\{x, y, w\}$ (w is an arbitrary element) with distance

$$\rho(x, y) = d(x, y) \text{ and } \rho(x, w) = \rho(y, w) = \frac{1}{2} d(x, y).$$

The identity mapping $I : Y \to M$ is nonexpansive, so by injectivity of M it has a nonexpansive extension $\widetilde{I} : X \to M$. Therefore,

$$\begin{aligned} d(x, y) &\leq d(x, \widetilde{I}(w)) + d(\widetilde{I}(w), y) \leq \rho(x, w) + \rho(y, w) \\ &= \frac{1}{2} d(x, y) + \frac{1}{2} d(x, y) = d(x, y). \end{aligned}$$

Now

$$\begin{aligned} d(x, \widetilde{I}(w)) &= d(\widetilde{I}(x), \widetilde{I}(w)) \leq \rho(x, w) = \frac{1}{2} d(x, y), \\ \text{and similarly} \quad d(y, \widetilde{I}(w)) &\leq \frac{1}{2} d(x, y). \end{aligned}$$

Also,

$$\begin{aligned} d(x, y) &\leq d(x, \widetilde{I}(w)) + d(\widetilde{I}(w), y) \leq d(x, \widetilde{I}(w)) + \frac{1}{2} d(x, y), \\ \text{or} \quad \frac{1}{2} d(x, y) &\leq d(x, \widetilde{I}(w)). \\ \text{Similarly,} \quad \frac{1}{2} d(x, y) &\leq d(y, \widetilde{I}(w)). \end{aligned}$$

Thus

$$d(x, \widetilde{I}(w)) = d(y, \widetilde{I}(w)) = \frac{1}{2} d(x, y).$$

and this proves that M is $M^{\oplus}$-convex.

Now we show that M has the binary ball intersection property. Suppose $\{B[x_i, r_i]\}_{i \in I}$ is a family of closed balls of M each two of which have nonempty intersection. Let Y be the subspace of M consisting of the points $\{x_i : i \in I\}$ and let $I_d : Y \to M$ be the identity mapping. As before, let $X = Y \cup \{w\}$, where w is an arbitrary element (of course, not in Y), and define the distance on X by extending d to

$$d(x_j, w) = \inf\{r : B[x_i, r] \subseteq B[x_j, r] \text{ for some } i \in I\}.$$

To see that this is a metric, one only need to check that for $j, k \in I$,

$$d(x_j, x_k) \leq d(x_j, w) + d(w, x_k).$$

[Here we shall use the binary ball intersection property. Note that since $B[x_j, r_j] \supseteq B[x_j, r_j]$,

$$d(x_j, w) \leq r_j \text{ for each } j \in I.$$

Also, either $d(x_j, w) = r_j$ or for some $i \neq j$, $B[x_i, r_i] \subseteq B[x_j, d(x_j, w)]$. In either case, given $j, k \in I$ there exist $i, n \in I$ such that

$$\begin{aligned} B[x_i, r_i] &\subseteq B[x_j, d(x_j, w)] \text{ and} \\ B[x_n, r_n] &\subseteq B[x_k, d(x_k, w)] \ \text{ (by definition of } d(x_j, w)). \end{aligned}$$

Since $B[x_i, r_i] \cap B[x_k, r_k] \neq \varnothing$, it must be the case that

$$B[x_j, d(x_j, w)] \cap B[x_k, d(x_k, w)] \neq \varnothing,$$

from which $d(x_j, x_k) \leq d(x_j, w) + d(w, x_k)$.]

Finally, since M is injective, I_d has a nonexpansive extension $\widetilde{I}_d : X \to M$. Then we have $\widetilde{I}_d(w) \in M$ and

$$d(x_j, \widetilde{I}_d(w)) = d(\widetilde{I}_d(x_j), \widetilde{I}_d(w) \leq d(x_j, w) \leq r_j \text{ for each } j \in I.$$

Therefore,

$$\widetilde{I}_d(w) \in \bigcap_{i \in I} B[x_i, r_i].$$

($\Longrightarrow$) Suppose (M, d) is hyperconvex, let Y be a metric space with $T : Y \to M$ nonexpansive, and suppose Y is a subspace of a metric space X. Consider the family $\mathcal{N}$ defined as follows:

$$\begin{aligned} \mathcal{N} &= \{(T_F, F) : F \text{ is a subspace of } X \text{ for which} Y \subseteq F \subseteq X \\ \text{and } T_F &: F \to M \text{ is nonexpansive with} T_F(Y) = T(y)\}. \end{aligned}$$

The family $\mathcal{N}$ can be partially ordered by setting

$$(T_F, F) \prec (T_G, G) \text{ iff } F \subseteq G \text{ and } T_G(z) = T_F(z) \text{ for each } z \in F.$$

Clearly $\mathcal{N} \neq \varnothing$ (since $(T, Y) \in \mathcal{N}$). Let $\{(T_{F_\alpha}, F_\alpha)\}_{\alpha \in A}$ is a chain (totally ordered subset) in $\mathcal{N}$. Take $G = \cup_{\alpha \in A} F_\alpha$ and define $T_G : G \to M$ by taking $T_G(x) = T_{F_\alpha}(x)$ for any α such that $x \in F_\alpha$. Then clearly

$$(T_{F_\alpha}, F_\alpha) \prec (T_G, G) \text{ for each } \alpha \in A,$$

so each chain in $\mathcal{N}$ is bounded above. By Zorn's Lemma $\mathcal{N}$ has a maximal element, say (T_H, H).

We assert $H = X$. If not, then there exist $z \in X$ such that $z \notin H$. Now let $H_1 = H \cup \{z\}$. We wish to define a nonexpansive extension of T_H on H_1 .We consider the family of balls $\{B[T_H(x); d(x, z)]\}_{x \in H}$. Note that for $x_1, x_2 \in H$,

$$d(T_H(x_1), T_H(x_2)) \leq d(x_1, x_2) \leq d(x_1, z) + d(x_2, z).$$

Since M is hyperconvex it follows that

$$\bigcap_{x \in H} B[T_H(x), d(x, z)] \neq \varnothing.$$

Select $w \in \bigcap\limits_{x \in H} B[T_H(x), d(x, z)]$ and define $T_{H_1} : H_1 \to M$ by taking $T_{H_1}(x) = T_H(x)$ if $x \in H$ and setting $T_{H_1}(z) = w$. Since

$$T_{H_1}(z) \in \bigcap_{x \in H} B[T_H(x); d(x, z)],$$

it follows that

$$d(T_{H_1}(x_1), T_{H_1}(z)) \leq d(x, z) \text{ for each } x \in H.$$

Therefore, T_{H_1} is nonexpansive. This implies $(T_H, H) \prec (T_{H_1}, H_1)$ with $H_1 \neq H$. contradicting the maximality of (T_H, H). Thus $H = X$ and $T : Y \to M$ has a nonexpansive extension. Hence (M, d) is injective. □

Corollary 5.1.6. ([Bail88], p. 14) *If H is hyperconvex and is contained in a metric space (M, d), then there is a nonexpansive retraction of M onto H.*

Proof. The identity mapping $i : H \to H$ is clearly nonexpansive:

$$d(i(x), i(y)) = d(x, y) \text{ for all } x, y \in H.$$

By above theorem, i can be extended to a nonexpansive mapping $i^* : M \to H$ such that $i^*(x) = i(x) = x$. So $i^* : M \to H$ is a nonexpansive retraction of M and it is clearly onto. □

5.2 Fixed Point Theorem for Nonexpansive Maps in Hyperconvex Spaces

The fixed point property (FPP) for nonexpansive mappings in hyperconvex spaces was first proved by Sine [Sine79] and Soardi [Soa79]. In this section we make a study of a fixed point theorem in hyperconvex spaces for such mappings. This theorem is due to Khamsi and Kirk in [KK01].

Definition: ([KK01], p. 36, 84) A bounded subset D of a metric space (M, d) is called **admissible** if

$$\begin{aligned} D &= cov(D) \\ &= \bigcap\{B \subseteq M : B \text{ is a closed ball and } B \supseteq D\}. \end{aligned}$$

Notation. The collection of all admissible subsets of a metric space M will be denoted by $\mathcal{A}(M)$.

Note. A set is admissible iff it can be written as the intersection of a family of closed balls in M. For this reason, the family $\mathcal{A}(M)$ is closed under arbitrary intersections. The admissible subsets of a hyperconvex space are interesting for another reason.

Theorem 5.2.1. ([KK01], p. 84) *Suppose (M, d) is a hyperconvex metric space. Then each set $D \in \mathcal{A}(M)$ is itself hyperconvex.*

Proof. Since D is admissible, we can write $D = \bigcap\limits_{i \in I} B[x_i, r_i]$, $x \in M$. Now let $\{B[x_\alpha, r_\alpha]\}_{\alpha \in A}$ be a family of closed balls centered at points $x_\alpha \in D$ for which $d(x_\alpha, x_\beta) \leq r_\alpha + r_\beta$, $\alpha, \beta \in A$. By the hyperconvexity of M,

$$\bigcap_{x \in A} B[x_\alpha, r_\alpha] \neq \varnothing.$$

It remains show that $\bigcap\limits_{x \in A} B[x_\alpha, r_\alpha] \subseteq D$. Now consider the family of closed balls

$$\{B[x_\alpha, r_\alpha] : \alpha \in A\} \cup \{B[x_i, r_i] : i \in I\ \}.$$

Then for $\alpha \in A$ and $i, j \in I$,

$$\begin{aligned} d(x_\alpha, x_i) &\leq r_i \leq r_\alpha + r_i \quad (\text{since } x_\alpha \in D) \\ \text{and } d(x_i, x_j) &\leq d(x_i, x_\alpha) + d(x_\alpha, x_j) \leq r_i + r_j, \end{aligned}$$

so it again follows from the hyperconvexity of M that

$$\left(\bigcap_{x \in A} B[x_\alpha, r_\alpha]\right) \bigcap \left(\bigcap_{i \in I} B[x_i, r_i]\right) = \left(\bigcap_{x \in A} B[x_\alpha, r_\alpha]\right) \bigcap D \neq \varnothing.$$

This proves that D is hyperconvex. $\square$

Theorem 5.2.2. ([Bail88], p. 14; [KK01], p. 84) *If (M,d) is a bounded hyperconvex metric space and if $T : M \to M$ is nonexpansive, then the fixed point set $Fix(T)$ of T in M is nonempty and hyperconvex.*

Proof. We first show that $Fix(T) \neq \varnothing$. This is another standard Zorn's Lemma approach. Let

$$\mathcal{F} = \{D \in \mathcal{A}(M) : D \neq \varnothing \text{ and } T : D \to D\}.$$

Then $\mathcal{F} \neq \varnothing$ since $M \in \mathcal{F}$ and $\mathcal{F}$ is partially ordered by reverse set-inclusion. ($A \geq B \Longleftrightarrow A \subseteq B$).

First note that if C is a chain in $\mathcal{F}$, then $\cap C \in \mathcal{A}(M)$.To see that $C \in \mathcal{F}$ we need only to show that $\cap C \neq \varnothing$.To see that this is true apply the definition of hyperconvexity. First index C, say $C = \{C_\alpha\}_{\alpha \in A}$. Each set C_α can be written as the intersection of closed balls in M, say $C_\alpha = \bigcap\limits_{i_\alpha \in I_\alpha} B[x_{i_\alpha}, r_{i_\alpha}]$.Suppose $C_\alpha \subseteq C_\beta$, and select $z \in C_\alpha$. Then for each $i_\alpha \in I_\alpha$, $i_\beta \in I_\beta$,

$$d(x_{i_\alpha}, x_{i_\beta}) \leq d(x_{i_\alpha}, z) + d(z, x_{i_\beta}) \leq r_{i_\alpha} + r_{i_\beta}.$$

Therefore by hyperconvexity of M,

$$\bigcap_{\alpha \in A} C_\alpha = \bigcap_{\alpha \in A} \left(\bigcap_{i_\alpha \in I_\alpha} B[x_{i_\alpha}, r_{i_\alpha}] \right) \neq \varnothing.$$

Since each chain in $\mathcal{F}$ is bounded above (by the intersection of the elements from the chain), $\mathcal{F}$ has a maximal element, which, of course is a minimal element relative to set inclusion. Call this minimal element D. Our strategy is to show that D consists of a single point.

Since

$$T(D) \subseteq \bigcap_{x \in M} B[x, r_x(T(D))] = cov(T(D))$$

and since $r_x(T(D)) \leq r_x(D)$, it must be the case that

$$\begin{aligned} cov(T(D)) &\subseteq \bigcap_{x \in M} B[x, r_x(T(D)] \subseteq \bigcap_{x \in M} B[x, r_x(D] = cov(D) = D; \\ \text{hence, } T\left(cov(T(D))\right) &\subseteq T(D) \subseteq cov(T(D)). \end{aligned}$$

This proves that $T : cov(T(D)) \to cov(T(D))$.

Since $cov(cov(T(D))) = cov(T(D))$, then $cov(T(D)) \in \mathcal{F}$, and, since $cov(T(D)) \subseteq D$ with D minimal, it must be the case that $D = cov(T(D))$.Thus

$$\begin{aligned} D &= \bigcap_{x \in M} B[x, r_x(T(D)], \\ \text{and so } \; r_x(D) &\leq r_x(T(D)) \text{ for each } x \in M. \end{aligned}$$

[Indeed if $y \in D$, then $y \in \bigcap_{x \in M} [Bx, r_x(T(D)]$ for each $x \in M$ so $d(y,x) \leq r_x(T(D)$; hence $r_x(D) \leq r_x(T(D))$. Therefore,

$$r_x(D) = r_x(T(D)),\ x \in M.$$

Now let $\delta = \delta(D)$ and let

$$C(D) = \bigcap_{x \in D} B[x, \frac{\delta}{2}].$$

We have already seen that $C(D) \neq \varnothing$ by hyperconvexity $\left(x, y \in D \Longrightarrow d(x,y) \leq \delta = \frac{\delta}{2} + \frac{\delta}{2}\right)$. What is important here is the fact that

$$D_1 = C(D) \cap D \neq \varnothing.$$

To see this we use the fact that D is admissible, say

$$D = \bigcap_{i \in I} B[x_i, r_i]$$

and consider the family

$$\mathcal{M} = \{B[x_i, r_i] : i \in I\} \cup \{B[x, \frac{\delta}{2}] : x \in D\}.$$

We have already seen that $d(x_i, x_j) \leq r_i + r_j$ for $i, j \in I$ and that $d(x,y) \leq \frac{\delta}{2} + \frac{\delta}{2}$ for $x, y \in D$. But also if $i \in I$ and $x \in D$, then by taking $z \in D$ we see that

$$d(x_i, x) \leq d(x_i, z) + d(z, x) \leq r_i + \frac{\delta}{2}.$$

So by hyperconvexity we conclude

$$D_1 = \cap \mathcal{M} \neq \varnothing.$$

Now let $z \in D_1$.

$$r_{T(z)}(T(D)) = \sup\{d(T(z), y) : y \in T(D)\}.$$

If $y \in T(D)$.Then $\exists\ p \in D$ such that$T(p) = y$ .Thus

$$\begin{aligned} r_{T(z)}(T(D)) &= \sup_{p \in D} d(T(z), T(p)) \\ r_{T(z)}(T(D)) &\leq \sup_{p \in D} d(z, p) \quad \text{(since T is nonexpansive).} \\ \text{Hence } r_{T(z)}(T(D)) &\leq r_z(D). \end{aligned}$$

However, we have already seen that

$$r_{T(z)}(T(D)) = r_{T(z)}(D),$$

and since $r_z(D) = \frac{\delta}{2}$ we obtain $r_{T(z)}(D) \le \frac{\delta}{2}$ so $T(z) \in C(D)$ and since $T(z) \in D$, we have shown $T(z) \in D_1$, that is, $T : D_1 \to D_1$.Since D_1 is admissible ($D_1 = cov(D_1)$), this proves that $D_1 \in \mathcal{F}$ and since $D_1 \subseteq D$ so by minimality of D we have $D_1 = D$. But $\delta(D_1) = \frac{\delta}{2}$ and $\delta(D) = \delta$. This can only happen if $\delta = 0$.We conclude, therefore, that D consists of a single point which of course must be a fixed point of T. We have proved the first part of the theorem, namely,that $Fix(T) \neq \varnothing$.

We now show that $Fix(T)$ is hyperconvex. Let $\{B[x_i, r_i]\}_{i\in I}$ be a family of closed balls centered at points $x_i \in Fix(T)$ with the property $d(x_i, x_j) \le r_i + r_j$ .Since M is hyperconvex the set

$$B = \bigcap_{i\in I} B_i[x_i, r_i] \neq \varnothing.$$

Since B is admissible, B is itself hyperconvex. Also, if $z \in B$ then for each $i \in I$,

$$d(T(z), x_i) = d(T(z), T(x_i)) \le d(z, x_i) \le r_i.$$

This proves that $T(z) \in B[x_i; r_i]$, that is, $T : B \to B$ so $B \in \mathcal{F}$ and since $B \subseteq D$ so by minimality of D we have $B = D$. Therefore, we can conclude from what we have shown above that T has a fixed point in B. Thus

$$\left(\bigcap_{i\in I} B_i[x_i, r_i]\right) \bigcap Fix(T) \neq \varnothing.$$

Hence $\bigcap_{i\in I} B_i[x_i; r_i] \subseteq Fix(T)$.This proves that $Fix(T)$ is hyperconvex. □

5.3 CAT(0) spaces and Fixed Points for Quasi-nonexpansive Maps

The notion of CAT(0) space is due to M. Gromov (see, e.g., [BH99]). This section contains the definition of CAT(0) spaces and its various properties, as given by Dhompongsa and Panyanak in [DhPa08]. Further, we present the connection between fixed point sets and closed convex subsets in CAT(0) space, as introduced by Chaoha and Phon-on in [CP06].

A metric space X is a CAT(0) space if it is geodesically connected, and if every geodesic triangle in X is at least as 'thin' as its comparison triangle in the Euclidean plane. The precise definition is given below. For a thorough discussion of these spaces and of the fundamental role they play in various branches of mathematics, see Bridson and Haefliger [BH99] or Burago, et al. [BBI01]. We note in particular that the complex Hilbert ball with a hyperbolic metric (see [GR84]; also inequality (4.2) of [RS90] and subsequent comments) is a CAT(0) space.

Definition: ([DhPa08], p. 2572) Let (X, d) be a metric space. A **geodesic path** joining $x \in X$ to $y \in X$ or, more briefly, a geodesic from x to y is a map c from a closed interval $[0, \ell] \subseteq \mathbb{R}$ to X such that $c(0) = x, c(\ell) = y$ and

$$d(c(t_1), c(t_2)) = |t_1 - t_2| \text{ for all } t_1, t_2 \in [0, \ell].$$

In particular, c is an isometry and $d(x, y) = \ell$.The image $\alpha = c([0, \ell])$ of c in X is called a **geodesic (or metric) segment** joining x and y. When unique, this geodesic is denoted by $[x, y]_c$ (or in short, $[x, y]$). Thus

$$[x, y]_c = \{c(\lambda) : \lambda \in [0, \ell] \text{ and } c(0) = x, c(\ell) = y\}.$$

Definition: ([DhPa08], p. 2573) The space (X, d) is said to be a **geodesic metric space** if every two points of X are joined by a geodesic, and X is said to be **uniquely geodesic** if there is exactly one geodesic joining x and y for each $x, y \in X$.

Definition: ([DhPa08], p. 2573) A subset Y of geodesic metric space (X, d) is said to be **g-convex** if, for each pair $x, y \in Y$, $[x, y]_c \subseteq Y$.

Definition: ([DhPa08], p. 2573) A **geodesic triangle** $\triangle\ (p, q, r)$ in a geodesic metric space (X, d) consists of three points p, q, r in X (the vertices of $\triangle$) and a geodesic segment between each pair of vertices (the edges of $\triangle$) .

Definition: ([BH99], p. 158; [DhPa08], p. 2573) A **comparison triangle** for geodesic triangle $\triangle\ (p, q, r)$ in (X, d) is a triangle $\overline{\triangle}(p, q, r) := \triangle(\overline{p}, \overline{q}, \overline{r})$ in the Euclidean plane $\mathbb{E}^2$ such that

$$d_{\mathbb{E}^2}(\overline{p}, \overline{q}) = d(p, q), d_{\mathbb{E}^2}(\overline{q}, \overline{r}) = d(q, r) \text{ and } d_{\mathbb{E}^2}(\overline{p}, \overline{r}) = d(p, r).$$

A point $\overline{x}$, belongs to a geodesic segment with endpoints $\overline{p}, \overline{q}$, is called a **comparison point** for x, belongs to a geodesic segment with endpoints p, q, if

$$d(q, x) = d_{\mathbb{E}^2}(\overline{q}, \overline{x}).$$

Definition: ([DhPa08], p. 2573) A geodesic metric space is said to be a **CAT(0)-space** if all geodesic triangles satisfy the following comparison axiom.

CAT(0) : Let $\triangle$ be a geodesic triangle in X, and let $\overline{\triangle}$ be a comparison triangle for $\triangle$. Then $\triangle$ is said to satisfy the CAT(0) inequality if for all $p, q \in \triangle$ and all comparison points $\overline{p}, \overline{q} \in \overline{\triangle}$,

$$d(p, q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}). \tag{C-I}$$

If x, y, z are points in a CAT(0) space and if m is the midpoint of the segment $[x, y]_c$, then the CAT(0) inequality (C-I) implies

$$d(z, m)^2 \leq \frac{1}{2}d(z, x)^2 + \frac{1}{2}d(z, y)^2 - \frac{1}{4}d(x, y)^2. \tag{CN}$$

This is the (CN) inequality of Bruhat and Tits [BT72].

Remark. (cf. [BH99], p. 163) *A geodesic space is a CAT*(0) *space iff it satisfies the (CN) inequality.*

As we see below, the parallelogram law in normed vector spaces is equivalent to the (CN) "equality", hence it satisfies the (CN) inequality.

Remark. Every Hilbert space $(X,<,>)$ satisfies the (CN) inequality.

Proof. Let $x,y,z \in X$, and let $m = \frac{x+y}{2}$, the mid point of x and y. Let $p = \frac{z-x}{2}$, $q = \frac{z-y}{2}$. By the parallelogram law:

$$\begin{aligned} ||p+q||^2 + ||p-q||^2 &= 2||p||^2 + 2||q||^2 \\ \text{or, } ||\frac{z-x}{2} + \frac{z-y}{2}||^2 + ||\frac{z-x}{2} - \frac{z-y}{2}||^2 &= 2||\frac{z-x}{2}||^2 + 2||\frac{z-y}{2}||^2 \\ \text{or } ||z - \frac{x+y}{2}||^2 + ||\frac{y-x}{2}||^2 &= \frac{2}{4}||z-x||^2 + \frac{2}{4}||z-y||^2, \\ \text{or } ||z-m||^2 &= \frac{1}{2}||z-x||^2 + \frac{1}{2}||z-y||^2 - \frac{1}{4}||x-y||^2, \\ \text{or } \ d(z,m)^2 &= \frac{1}{2}d(z,x)^2 + \frac{1}{2}d(z,y)^2 - \frac{1}{4}d(x,y)^2. \end{aligned}$$

Thus (CN) inequality holds. □

We now collect some elementary facts about CAT(0) spaces.

Lemma 5.3.1. **(**[DhPa08], p. 2573**)** *Let* (X,d) *be a CAT*(0) *space. Then*

(i) (X,d) *is uniquely geodesic* ([BH99], p. 160).

(ii) ([Kir04], Lemma 3) *Let* p,x,y *be points of* X*, let* $\alpha \in [0,1]$*, and let* m_1 *and* m_2 *denote, respectively, the points of* $[p,x]_c$ *and* $[p,y]_c$ *satisfying*

$$d(p,m_1) = \alpha d(p,x) \text{ and } d(p,m_2) = \alpha d(p,y).$$

Then

$$d(m_1,m_2) \leq \alpha d(x,y). \tag{1}$$

(iii) *Let* $x,y \in X$*,* $x \neq y$ *and* $z,w \in [x,y]_c$ *such that* $d(x,z) = d(x,w)$*. Then* $z = w$*.*

(iv) *Let* $x,y \in X$*. For each* $t \in [0,1]$*, there exists a unique point* $z \in [x,y]_c$ *such that*

$$d(x,z) = td(x,y) \textit{ and } d(y,z) = (1-t)d(x,y). \tag{2}$$

Proof. (i) Let $p,q \in (X,d)$. Consider two distinct geodesic segments σ,τ with same end points. Call these end points p and q. Let $r_1 \in \sigma$ and $r_2 \in \tau$ such that

$$d(p,r_1) = d(p,r_2).$$

Let σ' and σ'' be the two geodesic segments, with end points p,r_1 and r_1,q respectively, inside σ. Now let $\overline{\triangle}$ be a comparison triangle in $\mathbb{E}^2$ for geodesic triangle with vertices

p, r_1, q and geodesic segments τ, σ' and σ''. Suppose that comparison point for $r_1 \in \sigma$ is $\overline{r_1}$ such that

$$d(p, r_1) = d_{\mathbb{E}^2}(\overline{p}, \overline{r_1}),$$

and the comparison point for $r_2 \in \tau$ is $\overline{r_2}$ such that

$$d(p, r_2) = d_{\mathbb{E}^2}(\overline{p}, \overline{r_2})$$

But $\mathbb{E}^2$ is a uniquely geodesic metric space [BH99]. Then $\overline{r_1}, \overline{r_2} \in [\overline{p}, \overline{q}]_c$ and $d_{\mathbb{E}^2}(\overline{p}, \overline{r_1}) = d_{\mathbb{E}^2}(\overline{p}, \overline{r_2})$ (since $d(p, r_1) = d(p, r_2)$). It is obvious that $\overline{r_1} = \overline{r_2}$. [Indeed, since $\mathbb{E}^2$ is a geodesic metric space, let c be the geodesic path joining $\overline{p}$ and $\overline{q}$ and let $\ell = d_{\mathbb{E}^2}(\overline{p}, \overline{q})$. Since $\overline{r_1}, \overline{r_2} \in [\overline{p}, \overline{q}]_c$, there are $t_1, t_2 \in [0, \ell]$ such that $c(t_1) = \overline{r_1}$ and $c(t_2) = \overline{r_2}$. Thus

$$d_{\mathbb{E}^2}(\overline{p}, \overline{r_1}) = d_{\mathbb{E}^2}(c(0), c(t_1)) = |0 - t_1| = t_1,$$

and similarly $d_{\mathbb{E}^2}(\overline{p}, \overline{r_2}) = t_2$. Since $d_{\mathbb{E}^2}(\overline{p}, \overline{r_1}) = d_{\mathbb{E}^2}(\overline{p}, \overline{r_2})$, we have $t_1 = t_2$. That is $\overline{r_1} = \overline{r_2}$]. The CAT(0) inequality implies that

$$d(r_1, r_2) \leq d_{\mathbb{E}^2}(\overline{r_1}, \overline{r_2}).$$

Thus $d(r_1, r_2) = 0$, hence $r_1 = r_2$.

(ii) This is ([Kir04], Lemma 3).

(iii) Let c be the geodesic path joining x and y and let $\ell = d(x, y)$. Since $z, w \in [x, y]_c$, there are $t_1, t_2 \in [0, \ell]$ such that $c(t_1) = z$ and $c(t_2) = w$. Thus,

$$\begin{aligned} d(x, z) &= d(c(0), c(t_1)) = |0 - t_1| = t_1, \\ d(x, w) &= d(c(0), c(t_2)) = |0 - t_2| = t_2. \end{aligned}$$

Since $d(x, z) = d(x, w)$, we have $t_1 = t_2$. That is $z = w$.

(iv) If $x = y$, then the conclusion is obvious. Suppose that $x \neq y$. Take $z = c(t\ell)$. Thus $z \in [x, y]_c$ and

$$\begin{aligned} d(x, z) &= d(c(0), c(t\ell)) = |0 - t.\ell| = t\ell = td(x, y), \\ d(y, z) &= d(c(\ell), c(t\ell)) = |\ell - t.\ell| = (1 - t)\ell = (1 - t)d(x, y). \end{aligned}$$

The uniqueness of z follows from (iii). $\square$

For convenience, from now on we will use the notation $tx \oplus (1 - t)y$ for the unique point satisfying (2). By using this notation, the following is easy to verify.

Remark. Let (X, d) be a CAT(0) space and let $x, y \in X$ such that $x \neq y$ and $s, t \in [0, 1]$. Then

$$tx \oplus (1 - t)y = sx \oplus (1 - s)y \text{ iff } s = t.$$

Lemma 5.3.2. ([DhPa08], p. 2573) *Let (X, d) be a CAT(0) space and let $x, y \in X$ such that $x \neq y$. Then*

(i) $[x, y]_c = \{tx \oplus (1-t)y : t \in [0, 1]\}$.

(ii) $d(x, z) + d(z, y) = d(x, y)$ *iff* $z \in [x, y]_c$.

(iii) *The mapping* $f : [0, 1] \to [x, y]_c$, $f(t) = tx \oplus (1-t)y$ *is continuous and bijective.*

Proof. (i) The inclusion $\supseteq$ follows from definition. [In fact, by definition, for any $t \in [0, 1]$, $tx \oplus (1-t)y \in [x, y]_c$.]

For the converse inclusion $\subseteq$, let $z \in [x, y]_c$ and $1 - t = \frac{d(x,z)}{d(x,y)}$ so

$$t = 1 - \frac{d(x, z)}{d(x, y)} = \frac{d(y, z)}{d(x, y)}$$

Thus $z = tx \oplus (1-t)y$.

(ii) ($\Longrightarrow$) [CP06] Let $\triangle(\overline{x}, \overline{y}, \overline{z})$ be the comparison triangle in $\mathbb{E}^2$ of the geodesic triangle $\triangle(x, y, z)$, and $w \in [x, y]_c$ be such that $d(x, w) = d(x, z)$. By the above assumption,

$$d_{\mathbb{E}^2}(\overline{x}, \overline{z}) + d_{\mathbb{E}^2}(\overline{y}, \overline{z}) = d_{\mathbb{E}^2}(\overline{x}, \overline{y})$$

it follows that $[\overline{x}, \overline{y}]_c$ is simply a straight line with the point $\overline{z}$ in between; i.e., $\overline{z} \in [\overline{x}, \overline{y}]_c$. Moreover, $d(x, z) = d_{\mathbb{E}^2}(\overline{x}, \overline{z})$ and $d(x, w) = d_{\mathbb{E}^2}(\overline{x}, \overline{w})$ (by definition of comparison points). Since $d(x, w) = d(x, z)$,

$$d_{\mathbb{E}^2}(\overline{x}, \overline{w}) = d(x, w) = d(x, z) = d_{\mathbb{E}^2}(\overline{x}, \overline{z}).$$

Hence $d_{\mathbb{E}^2}(\overline{x}, \overline{w}) = d_{\mathbb{E}^2}(\overline{x}, \overline{z})$. Since $\mathbb{E}^2$ is uniquely geodesic, by the same argument of Lemma 5.3.1(iii), we get $\overline{w} = \overline{z}$. Then, by the CAT(0) inequality, we have

$$d(w, z) \leq d_{\mathbb{E}^2}(\overline{w}, \overline{z}) = 0,$$

which implies $z = w \in [x, y]_c$.

($\Longleftarrow$) let $z \in [x, y]_c$. By (i), there exists $t \in [0, 1]$ such that $z = tx \oplus (1-t)y$, consequently

$$d(x, z) + d(z, y) = td(x, y) + (1-t)d(x, y) = d(x, y).$$

(iii) Clearly f is well defined (by previous remark) and bijective (one-one (by remark), onto (by i)). Let c be the geodesic path joining x and y and let $\ell = d(x, y)$. Since $c : [0, \ell] \to [x, y]_c$ is continuous (because c is isometry) and the map $b : [0, 1] \to [0, \ell]$, defined by $b(t) = t\ell \ \forall \ t \in [0, 1]$ is also continuous, it follows that $g = c \circ b : [0, 1] \to [x, y]_c$, defined by $g(t) = (c \circ b)(t) = c(t\ell)$ is continuous. Now we want to show that $f = g$. Put $z = c(t\ell)$ for any $t \in [0, 1]$. Thus $z \in [x, y]_c$. With the same argument of Lemma 5.3.1(iv) and by definition of f we get $z = f(t)$ so $g(t) = f(t)$ for any $t \in [0, 1]$. Hence $f = g$. Therefore f is continuous.

Lemma 5.3.3. ([DhPa08], p. 2574; [PhSu11], p. 5) *Let X be a $CAT(0)$ space. Then*

$$d(tx \oplus (1-t)y, z) \leq td(x,z) + (1-t)d(y,z) \tag{3}$$

for all $x, y, z \in X$ and $t \in [0,1]$.

Proof. Let $x, y, z \in X$ and $t \in [0,1]$. Suppose that $d(z,y) \leq d(z,x)$. Let $u = tx \oplus (1-t)y$ and let x_0 be the point of $[z,x]_c$ such that $d(z,x_0) = d(z,y)$. Put $v = tx_0 \oplus (1-t)y$ and $w = tx_0 \oplus (1-t)z$. Since $v = tx_0 \oplus (1-t)y$, then $v \in [x_0,y]_c = [y,x_0]_c$ (since X is uniquely geodesic) and

$$d(y,v) = td(x_0,y) \text{ and } d(x_0,v) = (1-t)d(x_0,y).$$

Since $w = tx_0 \oplus (1-t)z$, $w \in [x_0,z]_c = [z,x_0]_c$ and

$$d(z,w) = td(x_0,z) \text{ and } d(x_0,w) = (1-t)d(x_0,z).$$

Also, since $u = tx \oplus (1-t)y$, $u \in [x,y]_c = [y,x]_c$ and

$$d(y,u) = td(x,y) \text{ and } d(x,u) = (1-t)d(x,y).$$

By Lemma 5.3.1(ii), we have

$$d(w,v) \leq (1-t)d(z,y) \text{ and } d(u,v) \leq td(x,x_0). \tag{*}$$

Now

$$\begin{aligned} d(z,v) &\leq d(z,w) + d(w,v) \\ &\leq td(x_0,z) + (1-t)d(z,y) \\ &= d(z,y) \text{ (since } d(z,x_0) = d(z,y)). \end{aligned}$$

Suppose that (3) does not hold. Then

$$\begin{aligned} td(z,x) + (1-t)d(z,y) &< d(z,u) \leq d(z,v) + d(v,u) \\ &\leq d(z,y) + d(v,u). \end{aligned} \tag{**}$$

Since $x_0 \in [z,x]_c$, by Lemma 5.3.2(ii),

$$\begin{aligned} d(z,x_0) + d(x_0,x) &= d(z,x), \\ \text{or } d(z,y) + d(x_0,x) &= d(z,x) \text{ (since } d(z,x_0) = d(z,y)). \end{aligned}$$

From $(**)$ we get

$$d(u,v) > t[d(z,x) - d(z,y)] = td(x,x_0).$$

This contradicts $(*)$. Hence

$$d(u,z) \leq td(x,z) + (1-t)d(y,z). \ \square$$

Corollary 5.3.4. ([PhSu11], p. 5) *$CAT(0)$ spaces are WCM spaces.*

Proof. This is obvious from above lemma by taking $W(x,y,t) := tx \oplus (1-t)y$ for all $x, y \in X$ and $t \in [0,1]$.

Remark. In Lemma 5.3.1(iii), we can rewrite it as follow: let $x, y \in X$. For each $t \in [0,1]$, there exists a unique point $W(x,y,t) \in [x,y]_c$.

Lemma 5.3.5. ([PhSu11], p. 5) *$CAT(0)$ spaces are uniformly WCM spaces.*

Proof. As above, a CAT(0) space is aWCM space.

To prove "uniformily" part, let $r > 0$, $\varepsilon \in (0,2]$, and $x, y, z \in X$ be such that $d(x,z) \leq r$, $d(y,z) \leq r$, $d(x,y) \geq \varepsilon.r$. It is obvious that$W(x,y,\frac{1}{2})$ is the midpoint of x and y. Applying (CN inequality) we get that

$$\begin{aligned} d(W(x,y,\frac{1}{2}),z) &\leq \sqrt{\frac{1}{2}d(x,z)^2 + \frac{1}{2}d(y,z)^2 - \frac{1}{4}d(x,y)^2} \\ &\leq \sqrt{\frac{1}{2}r^2 + \frac{1}{2}r^2 - \frac{1}{4}\varepsilon^2 r^2} \\ &= \sqrt{1 - \frac{\varepsilon^2}{4}}.r \leq (1 - \frac{\varepsilon^2}{8})r. \end{aligned}$$

where $\delta = \frac{\varepsilon^2}{8}$. $\square$

There are several ways to construct news examples of CAT(0) spaces from known ones.

Lemma 5.3.6. ([CP06], p. 984) *A g-convex subset of a $CAT(0)$ space is itself a $CAT(0)$ space when endowed with the induced metric.*

Proof. Let (X,d) be a CAT(0) space. Let Y be a g-convex subset of (X,d) with induced metric. Since$Y$ is g-convex, every geodesic segment joining any two of its points must be in Y. Now let $\triangle$ be a geodesic triangle in Y and $\overline{\triangle}$ be a comparison triangle for $\triangle$. Let $p, q \in Y$. Since (X,d) is a CAT(0) space and $Y \subseteq X$, then for all $p, q \in \triangle$ and all $\overline{p}$,$\overline{q} \in \overline{\triangle}$ we have

$$d(p,q) \leq d_{\mathbb{E}^2}(\overline{p}, \overline{q}).$$

i.e. $\triangle$ satisfies the CAT(0) inequality. Thus Y is a CAT(0) space.

Fixed Points

Recall that a mapping $T : X \to X$ is called a **quasi-nonexpansive** if it satisfies

$$d(T(x),u) \leq d(x,u) \text{ for all } x \in X \text{ and } u \in Fix(T). \tag{Q}$$

Lemma 5.3.7.*Let (X, d) be a metric space and $T : X \to X$ a quasi-nonexpansive map. Then $Fix(T)$ is closed.*

Proof. See Section 2.1.

Theorem 5.3.8. **(**[CP06], p. 985**)** *Let (X, d) be a g-convex subset of a $CAT(0)$ space and $T : X \to X$ a quasi-nonexpansive map whose fixed point set is nonempty. Then $Fix(T)$ is closed and g-convex.*

Proof. By above Lemma, $Fix(T)$ is closed.

We now show that $Fix(T)$ is convex. Since (X, d) is convex in a CAT(0) space, it is also a CAT(0) space. To show that $Fix(T)$ is g-convex, we need to show that, for any $x, y \in Fix(T)$, $[x, y]_c \subseteq Fix(T)$. Let $z \in [x, y]_c$. Since T is quasi-nonexpansive, we have $d(x, T(z)) \leq d(x, z)$ and $d(y, T(z)) \leq d(y, z)$. Hence, we obtain the inequalities

$$\begin{aligned} d(x, y) &\leq d(x, T(z)) + d(T(z), y) \\ &\leq d(x, z) + d(z, y) \\ &= d(x, y) \text{ (since } z \in [x, y]_c) \end{aligned}$$

which implies $d(x, T(z)) + d(T(z), y) = d(x, y)$. In fact, it is not difficult to see that $d(x, z) = d(x, T(z))$ and $d(z, y) = d(T(z), y)$. [For if $d(x, z) < d(x, T(z))$ or $d(y, z) < d(y, T(z))$, the above inequalities will give

$$d(x, y) \leq d(x, T(z)) + d(T(z), y) < d(x, z) + d(z, y) = d(x, y).$$

which is a contradiction.] Now, by Lemma 5.3.2(ii), we have $T(z) \in [x, y]_c$, and because $z, T(z) \in [x, y]_c$ satisfying $d(x, z) = d(x, T(z))$ hence by Lemma 5.3.1(iii) $z = T(z)$. Therefore, $[x, y]_c \subseteq Fix(T)$; i.e., $Fix(T)$ is g-convex. $\square$

5.4 $\mathbb{R}$-trees Metric Spaces vs Hyperconvex and CAT(0) Spaces

The notion of $\mathbb{R}$-trees was introduced by J. Tits in [Tit77]. In this section, we give connection between $\mathbb{R}$-trees metric spaces with hyerconvex and CAT(0) spaces, as given in [Kir98] and [BH99].

Definition: A metric space (X, d) is called an **$\mathbb{R}$-tree** if

(i) (X, d) is a geodesic space;

(ii) if $[x, y]_c \cap [x, z]_c = \{x\}$, then $[y, z]_c = [x, y]_c \cup [x, z]_c$;

(iii) $\forall\ x, y, z \in X : \exists w \in X : [x, y]_c \cap [x, z]_c = [x, w]_c$.

Examples. [Mart09] (1) The real line $\mathbb{R}$ is an $\mathbb{R}$-tree.

(2) $\mathbb{R}^2$ with the Euclidian metric is not an $\mathbb{R}$-tree because it contains metric (geodesic) triangles and thus violates Condition (ii) of the definition of $\mathbb{R}$-tree.

(3) The two sided comb metric on $\mathbb{R}^2$, which is given by

$$d((x_1,y_1),(x_2,y_2)) := \begin{cases} |y_1 - y_2| & \text{if } x_1 = x_2, \\ |y_1| + |x_1 - x_2| + |y_2| & \text{if } x_1 \neq x_2, \end{cases}$$

makes $\mathbb{R}^2$ into an $\mathbb{R}$-tree.

Lemma 5.4.1. ([Mart09], p. 2) *An $\mathbb{R}$-tree is geodesically unique.*

Proof. Consider two distinct geodesic segments σ, τ with the same endpoints. Argue by contradiction. Since the intersection $\sigma \cap \tau$ is compact there exists subgeodesics σ' and τ' which intersect at and only at their endpoints. Call these endpoints x and y.

Consider the geodesic σ' and $[y,z]_c$ inside τ'. These intersect in $\{y\}$. By (ii), $\sigma' \cup [y,z]_c = [x,z]_c$ and thus $d(x,z) = d(x,y) + d(y,z)$. This gives a contradiction if z is chosen such that $d(x,z) < d(x,y)$. Hence $\sigma = \tau$. $\square$

Lemma 5.4.2. ([Mart09], p. 3) *If (X,d) is geodesically unique, then (X,d) satisfies property (ii) of the definition of $\mathbb{R}$-tree.*

Proof. Let $[x,y]_c$ and $[x,z]_c$ be geodesics. Then the endpoints of the intersection $[x,y]_c \cap [x,z]_c$ are x and some $w \in [x,y]_c \cap [x,z]_c$. Now geodesics are unique. $\square$

Lemma 5.4.3. ([Mart09], p. 3) *Let (X,d) be an $\mathbb{R}$-tree. Let $x,y,z \in X$ and suppose that*

$$[x,y]_c \cap [x,z]_c = [x,w]_c.$$

Then:

(1) $[y,w]_c \cap [w,z]_c = \{w\}, [y,z]_c = [y,w]_c \cup [w,z]_c$ *and* $[x,y]_c \cap [w,z]_c = \{w\}$;

(2) $d(x,w) = \frac{1}{2}(d(x,y) + d(x,z) - d(y,z))$;

(3) *The point w depends only on x,y,z and not on their order.*

Proof. (1) It is enough to show that $[x,y]_c \cap [w,z]_c = \{w\}$. Therefore let

$$u \in [x,y]_c \cap [w,z]_c \subseteq [x,y]_c \cap [x,z]_c = [x,w]_c$$

(since $[x,w]_c \subseteq [x,z]_c$). Then $d(u,x) \leq d(w,x)$. But we have also that $u \in [w,z]_c$. Certainly $d(u,z) \leq d(w,z) \leq d(x,z)$ and thus

$$\begin{aligned} d(x,u) &= d(x,z) - d(u,z) \text{ (since } u \in [x,z]_c) \\ &\geq d(x,z) - d(z;w) \text{ (since } d(u,z) \leq d(w,z)) \\ &= d(x,w) \text{ (since } [z,w]_c \subseteq [x,z]_c \text{ and } w \in [x,w]_c \subseteq [x,z]_c) \end{aligned}$$

Hence $d(x,u) \geq d(x,w)$. and so $d(u,x) = d(x,w)$ (since $u \in [w,x]_c$) which implies that $u = w$.

(2)

$$
\begin{aligned}
d(y,z) &= d(y,w)+d(w,z)\\
&= [d(y,x)-d(x,w)]+[d(z,x)-d(x,w)]\\
&= -2d(x,w)+d(y,x)+d(x,z)
\end{aligned}
$$

Thus $d(x,w)=\frac{1}{2}[d(x,y)+d(x,z)-d(y,z)]$.

(3) Consider $[y,x]_c \cap [y,z]_c = [y,v]_c$ for some $v \in X$. We want to show that $v = w$ and for this it is enough to show that $d(y,v)=d(y,w)$. Then we get from part (2):

$$
\begin{aligned}
2d(y,v) &= d(y,x)+d(y,z)-d(x,z)\\
&= [d(y,w)+d(w,x)]+[d(y,w)+d(w,z)]-[d(x,w)+d(w,z)]\\
&= 2d(y,w). \ \square
\end{aligned}
$$

Recall that: (i) A bounded subset D of a metric space (M,d) is called **admissible** if

$$
\begin{aligned}
D &= cov(D)\\
&= \bigcap\{B \subseteq M : B \text{ is a closed ball and } B \supseteq A\}.
\end{aligned}
$$

(ii) $r(A)=\inf\{r_x(A) : x \in A\}$.

Recall that**:** $\mathcal{A}(M)$ is said to be **normal** if for each $D \in \mathcal{A}(M)$ for which $\delta(D) > 0$,we have $r(D) < \delta(D)$.

$\mathcal{A}(M)$ is said to be **uniformly normal** if there exists $c < 1$ such that for each $D \in \mathcal{A}(M)$ for which $\delta(D) > 0$, we have $r(D) \le c\ \delta(D)$.

Finally, $\mathcal{A}(M)$ is said to be **compact** [resp., **countably compact**] if every family [resp. countable family] of nonempty sets in $\mathcal{A}(M)$ which has the finite intersection property has nonempty intersection. (The intersection of such a family is necessarily also a member of $\mathcal{A}(M)$.)

Remark. Any uniformly normal is normal

Theorem 5.4.4. [Kham89] *Let M be a complete metric space for which $\mathcal{A}(M)$ is uniformly normal. Then $\mathcal{A}(M)$ is countably compact.*

Theorem 5.4.5. [KuLi96] *Let M be a metric space for which $\mathcal{A}(M)$ is countably compact and normal. Then $\mathcal{A}(M)$ is compact.*

Results. We begin with the following.

Theorem 5.4.6. ([Kir98], p.69) *If (M,d) is an ℝ-tree, then $\mathcal{A}(M)$ is uniformly normal.*

Proof. Let $\varepsilon \in (0,1)$. For $D \in \mathcal{A}(M)$ with $\delta := \delta(D) > 0$, select $u, v \in D$ such that $d(u,v) > (1-\varepsilon)\delta$, and let $x \in D$ be arbitrary. By (iii) of definition of ℝ-tree, there exists $w \in [u, v]_c$ such that $[u, v]_c \cap [u, x]_c = [u, w]_c$. In particular,

$$d(x, u) = d(x, w) + d(w, u) \text{ and } d(x, v) = d(x, w) + d(w, v).$$

Suppose m is the midpoint of $[u, v]_c$. If $w \in [u, m]_c$ then

$$\delta \geq [d(x, w) + d(w, m)] + d(m, v) > d(x, m) + \frac{1}{2}(1 - \varepsilon)\delta$$

and it follows that

$$d(x, m) \leq \frac{1}{2}(1 + \varepsilon)\delta$$

Similarly the same conclusion follows if $w \in [v, m]_c$.Thus

$$D \subseteq B[m, \frac{1}{2}(1 + \varepsilon)\delta].$$

Since any closed ball in M contains the segment joining any two of its points (this also is a simple consequence of (iii) of definition of $\mathbb{R}$-tree), and since $D \in \mathcal{A}(M)$,and since $D \in \mathcal{A}(M)$, we have $m \in D$. Therefore

$$r(D) \leq \frac{1}{2}(1 + \varepsilon)\delta \ \ (\text{since } r(D) = \inf_{x \in D} r_x(D)).$$

Since $\varepsilon > 0$ is arbitrary we conclude that $\mathcal{A}(M)$ is uniformly normal with constant $c = \frac{1}{2}$. □

Theorem 5.4.7. ([Kir98], p.69) *Let a metric space (M, d) be a complete $\mathbb{R}$-tree. Then M is hyperconvex.*

Proof. We first show that if $\{B[x_i, r_i] : i = 1, ..., n\}$ is an arbitrary finite collection of closed balls in an $\mathbb{R}$-tree M, any two of which intersect, then

$$\bigcap_{i=1}^{n} B[x_i, r_i] \neq \varnothing$$

We proceed by induction on n. The conclusion is trivial if $n = 2$. Suppose that for fixed $n \geq 2$ each family of n balls, any two of which intersect, has nonempty intersection, and suppose that any two balls of the family $\{B[x_i, r_i] : i = 1, ..., n + 1\}$ intersect. Then by the inductive hypothesis

$$S := \cap_{i=1}^{n} B[x_i, r_i] \neq \varnothing.$$

Now suppose

$$B[x_{n+1}, r_{n+1}] \cap S = \varnothing$$

and let $p \in S$. Then $p \notin B[x_{n+1}, r_{n+1}]$ so we have $d(x_{n+1}, p) > r_{n+1}$.Let t be the point of $[x_{n+1}, p]_c$ for which $d(x_{n+1}, t) = r_{n+1}$.(thus $t \in B[x_{n+1}, r_{n+1}]$), and let $i \in \{1, ..., n\}$.There are two cases:

(I) $t \notin [x_i, p]_c$. In this case

$$[x_i, t]_c \cap [x_{n+1}, t]_c = \{t\}$$

(since $t \in [x_{n+1}, p]_c$), so by (ii) of definition of ℝ-tree,

$$[x_i, t]_c \cup [x_{n+1}, t]_c = [x_i, x_{n+1}]_c,$$

then we have $t \in [x_{n+1}, x_i]_c$ and therefore t is the point of $B[x_{n+1}, r_{n+1}]$ (from above) nearest to x_i; hence $t \in B[x_i; r_i]$ by the binary intersection property.

(II) $t \in [x_i, p]_c$. In this case

$$d(x_i, t) \leq d(x_i, p) \leq r_i$$

so again $t \in B[x_i, r_i]$.

Therefore $t \in B[x_i, r_i]$ in either case. Since i is arbitrary so $t \in \cap_{i=1}^{n+1} B[x_i, r_i]$, completing the induction.

Now suppose M is a complete ℝ-tree. Since M is $M^{\oplus}$-convex, to see that M is hyperconvex it need only be shown that $\cap_{\alpha\in A} B[x_\alpha, r_\alpha] \neq \varnothing$ whenever $\{B[x_\alpha, r_\alpha]\}_{\alpha\in A}$ is any family of closed balls in M any two of which intersect. However, if any two balls in such a family intersect then [by what we have seen above] the family $\{B[x_\alpha, r_\alpha]\}_{\alpha\in A}$ has the finite intersection property.

Also,$\{B[x_\alpha, r_\alpha]\}_{\alpha\in A}$ is a subfamily of $\mathcal{A}(M)$ (since $B[x_\alpha, r_\alpha]$ is a closed ball), and by Theorem 5.4.6, $\mathcal{A}(M)$ is uniformly normal so it is normal. Now, since M is complete and uniformly normal $\mathcal{A}(M)$ is countably compact by Theorems 5.4.4. Since $\mathcal{A}(M)$ is normal, it is compact by Theorem 5.4.5, so any subfamily of $\mathcal{A}(M)$ which has the finite intersection property must have nonempty intersection. Therefore $\cap_{\alpha\in A} B[x_\alpha, r_\alpha] \neq \varnothing$ proving.

Theorem 5.4.8. ([BH99], p. 167**)** ℝ-trees are CAT(0) spaces.

5.5 Hyperbolic Spaces and T-Invariant Approximation for Quasi-nonexpansive Maps

Different notions of "hyperbolic spaces" ([Kir81-82], [GK83], [GK84], [RS90]) can be found in the literature. We deal with the setting of hyperbolic spaces, as introduced by Kohlenbach [Koh05] and Leustean in [Le07]. We present quasi-nonexpansive mappings satisfy a certain condition, namely, condition (K), as given by Laowang and Panyanak in [LaoPan13].

Definition: ([Koh05], p. 98; [Le07], p. 387) A **hyperbolic space** is a triple (X, d, W) where (X, d) is a metric space and $W : X \times X \times [0, 1] \to X$ is such that for all $x, y, z, u \in X$ and $\alpha, \beta \in [0, 1]$, we have

$(W_1) = (W)$: $d(z, W(x, y, \alpha)) \leq \alpha d(x, z) + (1-\alpha)d(z, y)$;
$(W_2) = (I)$: $d(W(x, y, \alpha), W(x, y, \beta)) = |\alpha - \beta|\, d(x, y)$;
$(W_3) = (C)$: $W(x, y, \alpha) = W(y, x, 1-\alpha)$;
$(W_4) = (S)$: $d(W(x, z, \alpha), W(y, u, \alpha)) \leq \alpha d(x, y) + (1-\alpha)d(z, u)$.

If $x, y \in X$, and $\alpha \in [0,1]$, then we use the notation $\alpha x \oplus (1-\alpha)y$ for $W(x, y, \alpha)$. It is easy to see that for any $x, y \in X$, and $\alpha \in [0,1]$, we have

$$\begin{aligned} d(x, \alpha x \oplus (1-\alpha)y) &= (1-\alpha)d(x, y), \\ d(y, \alpha x \oplus (1-\alpha)y) &= \alpha d(x, y). \end{aligned}$$

We shall denote by $[x, y]$ the set $\{\alpha x \oplus (1-\alpha)y : \alpha \in [0,1]\}$.

Definition: A nonempty subset C of a hyperbolic space X is said to be **convex** if $[x, y] \subseteq C$ for all $x, y \in C$.

Definition: [Le10] If C is a convex subset of a hyperbolic space X, then a function $f : C \to \mathbb{R}$ is said to be **convex** if

$$f(\lambda x \oplus (1-\lambda)y) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } x, y \in C, \lambda \in [0,1].$$

Remark. We can use this definition for WCM space, i.e.

$$f(W(x, y, \lambda) \leq \lambda f(x) + (1-\lambda)f(y) \quad \text{for all } x, y \in C, \lambda \in [0,1].$$

Corollary 5.5.1. ([Le07], p. 387) *$CAT(0)$ spaces are hyperbolic spaces.*

Proof. (Out line) As proved in Section 5.3, a CAT(0) space (X, d) is a WCM space so $(W1)$ is satisfied. Since CAT(0) space is uniformly WCM space (see section 5.3), $(W2 - W3)$ is satisfied (we already have proved that in Section 3.2 of Chapter 3). To prove that CAT(0) space satisfies $(W4)$, it suffices show that it satisfies this property,

$$d(W(x, y, \alpha), W(x, z, \alpha)) \leq (1-\alpha)d(y, z) \qquad (H_1)$$

for all $x, y, z \in X$ and $\alpha \in [0,1]$. [In fact, we already have proved that in Section 3.2 (Lemma 3.2.2) of Chapter 3, uniformly WCM spaces with property (H_1) satisfy $(W4)$]. Now, let $x, y, z \in X$ and $\alpha \in [0,1]$. Then$W(x, y, \alpha) = \alpha x \oplus (1-\alpha)y \in [x, y]$ and $W(x, z, \alpha) = \alpha x \oplus (1-\alpha)z \in [x, z]$. Then,

$$d(x, W(x, y, \alpha)) = d(x, \alpha x \oplus (1-\alpha)y) = (1-\alpha)d(x, y),$$

and

$$d(x, W(x, z, \alpha)) = d(x, \alpha x \oplus (1-\alpha)z) = (1-\alpha)d(x, z).$$

Thus by Lemma 5.3.1(ii) in Section 5.3,

$$d(W(x, y, \alpha), W(x, z, \alpha)) \leq (1-\alpha)d(y, z). \quad \square$$

Definition: The hyperbolic space (X, d, W) is called **uniformly convex** if for any $r > 0$, and $\varepsilon \in (0, 2]$ there exists a $\delta \in (0, 1]$ such that for all $x, y, z \in X$ with $d(x, z) \leq r$, $d(y, z) \leq r$, and $d(x, y) \geq r\ \varepsilon$, it is the case that

$$d(\frac{1}{2}x \oplus \frac{1}{2}y, z) \leq (1 - \delta)r.$$

Note. We remark that any normed space $(X, \|\cdot\|)$ is a hyperbolic space, with $\lambda x \oplus (1 - \lambda)y := \lambda x + (1 - \lambda)y$. Obviously, uniformly convex Banach spaces are uniformly convex hyperbolic spaces.

Theorem 5.5.2. ([Le07], p. 391) *Assume that (X, d) is a $CAT(0)$ space. Then (X, d) is a uniformly convex hyperbolic space.*

Proof. By using the same argument of Lemma 5.3.5 in Section 5.3.

From now on, X stands for uniformly convex hyperbolic space.

Proposition 5.5.3. ([KohLe07], Proposition 2.2) *The intersection of any decreasing sequence of nonempty bounded closed convex subsets of a uniformly convex hyperbolic space X is nonempty.*

Proposition 5.5.4. [Le10] *Let C be a nonempty closed convex subset of a uniformly convex hyperbolic space X, $f : C \to [0, \infty)$ be convex and lower semicontinuous. Assume moreover that for all sequences (x_n) in C,*

$$\lim_{n\to\infty} d(x_n, a) = \infty \text{ for some } a \in X \text{ implies } \lim_{n\to\infty} f(x_n) = \infty.$$

Then f attains its minimum on C. If, in addition,

$$f(\frac{1}{2}x \oplus \frac{1}{2}y) \leq \max\{f(x), f(y)\}$$

for all $x \neq y$, then f attains its minimum at exactly one point.

Proof. Let α be the infimum of f on C and define

$$C_n := \{x \in C : f(x) \leq \alpha + \frac{1}{n}\}$$

for all $n \in \mathbb{N}$.for all $n \in \mathbb{N}$. It is easy to see that C_n is closed (since f is lower semicontinuous), convex (since f is convex), nonempty (since $\alpha \leq f(x)$, $\forall\ x \in C$, take $\varepsilon = \frac{1}{n}$ then by definition of infimum $\exists$ a $y_n \in C$ such that

$$\alpha \leq f(y_n) \leq \alpha + \frac{1}{n}.$$

Hence $y_n \in C_n$) and the sequence $(C_n)_{n\in\mathbb{N}}$ is decreasing (let $y \in C_{n+1}$, then $f(y) \leq \alpha + \frac{1}{n+1} \leq \alpha + \frac{1}{n}$. Thus $y \in C_n$) we can apply Theorem 5.5.3 to the sequence $(C_n)_{n\in\mathbb{N}}$ to get the existence of $x^* \in \cap_{n\in\mathbb{N}} C_n$. It follows that $f(x^*) \leq \alpha + \frac{1}{n}$ for all $n \geq 1$, hence $f(x^*) \leq \alpha$. Since α is the infimum of f, we can conclude that $f(x^*) = \alpha$, that is f attains its minimum on C. The second part of the conclusion is immediate. If f attains its minimum at two points $x \neq y$, then $\frac{1}{2}x^* \oplus \frac{1}{2}y\ ^* \in C$, since C is convex, and

$$f(\frac{1}{2}x^* \oplus \frac{1}{2}y^*) < \max\{f(x^*), f(y^*)\} = \alpha,$$

which is a contradiction. □

Theorem 5.5.5. [Le10] *Every nonempty closed convex subset C of a uniformly convex hyperbolic space X is a Chebyshev set.*

Proof. Let $x \in X$ and define $f : C \to [0, \infty)$, $f(y) = d(x, y)$. Then f is continuous, convex (by definition of convex function), and for any sequence (y_n) in C, $\lim\limits_{n\to\infty} d(y_n, a) = \infty$ for some $a \in X$ implies $\lim\limits_{n\to\infty} f(y_n) = \infty$, since

$$f(y_n) = d(x, y_n) \geq d(y_n, a) - d(x, a).$$

Moreover, let $y \neq z \in C$ and denote

$$M := \max\{f(y), f(z)\} > 0.$$

Then

$$d(x, y), d(x, z) \leq M \text{ and } d(y, z) \geq \varepsilon.M,$$

where $\varepsilon := \frac{d(y,z)}{M}$ and $0 < \varepsilon \leq \frac{d(x,y)+d(x,z)}{M} \leq 2$. Hence by uniform convexity $\delta \in (0, 1]$ such that

$$d(\frac{1}{2}y \oplus \frac{1}{2}z, x) \leq (1 - \delta).M < M.$$

Thus, f satisfies all the hypotheses of Proposition 5.5.4, so we can apply it to conclude that f has a unique minimum. Hence, C is a Chebyshev set. □

Note. The hyperbolicity is stronger than W-convexity. In fact, hyperbolic space implies that WCM space.

Lemma 5.5.6. ([KaePa08], p. 7) *Let X be a uniformly convex hyperbolic space, and let $x, y, z \in X$ for which*

$$d(x, z) + d(z, y) = d(x, y). \tag{1}$$

Then $z \in [x, y]$.

Proof. Let $u \in x, y$ be such that $d(x, u) = d(x, z)$. Then

$$d(x, y) = d(x, u) + d(u, y) \text{ and also } d(z, y) = d(u, y) \text{ by } (1).$$

We will show that $z = u$. Suppose not, we let $v = (1/2)z \oplus (1/2)u$ and $r = d(x, u) = d(x, z)$. Since $d(z, u) > 0$, choose $\varepsilon > 0$ so that $d(z, u) > r.\varepsilon$. By the uniform convexity of X, there exists $\delta > 0$ such that

$$d(x, v) \leq r(1 - \delta) < r = d(x, z).$$

By using the same arguments, we can show that $d(y,v) < d(y,z)$. Therefore

$$d(x,y) \leq d(x,v) + d(y,v) < d(x,z) + d(y,z),$$

which is a contradiction. □

By using the above lemma with the proof of Theorem 5.3.8, we obtain the following result.

Theorem 5.5.7. ([KaePa08], p. 8) *Let C be a convex subset of a uniformly convex hyperbolic space X and $f : C \to C$ a quasi-nonexpansive mapping with $Fix(f) \neq \varnothing$. Then $Fix(f)$ is closed and convex.*

We now define a condition (K) which is related to the T-invariant approximation property. .

Definition: A single-valued mapping $T : C \to C$ is said to satisfy **condition (K)** if $Fix(T)$ is nonempty closed and convex, and for each $x \in Fix(T)$ and any closed convex subset A with $T(A) \subseteq A$, $P_A(x) \in Fix(T)$, hence $P_A(x) \cap Fix(T) \neq \varnothing$.

Theorem 5.5.8. ([LaoPan13], p. 3) *Let C be a nonempty convex subset of X. If $T : C \to C$ is a quasi-nonexpansive mapping, then T satisfies condition (K).*

Proof. $Fix(T)$ is closed and convex (by Theorem 5.5.7). Let $x \in Fix(T)$ and A be a closed convex subset of C with $T(A) \subseteq A$. Let $y \in A$ be such that $d(x,y) = d(x,A)$. Since T is quasi-nonexpansive,

$$d(x, T(y)) \leq d(x,y).$$

Thus, since $T(y) \in A$,

$$\begin{aligned} d(x,y) &= d(x,A) \leq d(x,T(y)) \leq d(x,y), \\ \text{or} \quad d(x,T(y)) &= d(x,y) = d(x,A). \end{aligned}$$

By the uniqueness of y (by Theorem 5.5.5), we have $T(y) = y$. Therefore T satisfies condition (K). □

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